HOMEWORK 2

DUE 21 APRIL 2017

Review Problems 1 and 3–6 in HW4 of 200B
(`http://www.math.ucsd.edu/~jmckerna/Teaching/16-17/Winter/200B/problems.html`).

1. Let R be a ring and M, N be R-modules. The functors $\mathrm{Hom}_R(M, -) : R\textbf{-mod} \longrightarrow R\textbf{-mod}$ and $\mathrm{Hom}_R(-, N) : R\textbf{-mod} \longrightarrow R\textbf{-mod}$ are left exact (for a given definition of exactness of contravariant functors). Does this still hold if we think of the two functors as $R\textbf{-mod} \longrightarrow \mathbb{Z}\textbf{-mod}$?

2. Write down explicitly the isomorphism $\mathrm{Hom}_R(M \otimes_R N, P) \longrightarrow \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$ and show that it is functorial, i.e. for each pair of R-module homomorphisms $f : M' \longrightarrow M$ and $g : P \longrightarrow P'$, and for any R-module N the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_R(M \otimes_R N, P) & \xrightarrow{\approx} & \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)) \\
\Big\downarrow {\scriptstyle g\circ(-)\circ(f\otimes 1_N)} & & \Big\downarrow {\scriptstyle g_*\circ(-)\circ f} \\
\mathrm{Hom}_R(M' \otimes_R N, P') & \xrightarrow[\approx]{} & \mathrm{Hom}_R(M', \mathrm{Hom}_R(N, P'))
\end{array}$$

is commutative. Here g_* denotes the pushforward of g.

3. Let $R = \mathbb{Z}[\sqrt{-6}] = \{a + b\sqrt{-6}; a, b, \in \mathbb{Z}\}$. Let $\mathfrak{a} = (2, \sqrt{-6})$ be the ideal of R generated by 2 and $\sqrt{-6}$.

(a) Show that $\mathfrak{a}$ is not a free R-module.

(b) Show that $\mathfrak{a}$ is a projective R-module.

4. Let G be a group. A (left) *G-module* is an abelian group M on which there is a G action which satisfies for all $m, m' \in M$ and $\sigma, \tau \in G$,

$$\begin{aligned}
1_G m &= m, \\
\sigma(\tau m) &= (\sigma\tau) m, \\
\sigma(m + m') &= \sigma m + \sigma m'.
\end{aligned}$$

That is, there is a group homomorphism $G \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(M) : \sigma \mapsto \sigma(\cdot)$. A morphism of G-modules $f : M \longrightarrow N$ is a group homomorphism which also satisfies $f(\sigma m) = \sigma f(m)$, for $m \in M$ and $\sigma \in G$. For a G-module M, the subgroup of *G-invariant* elements of M is

$$M^G := \{m \in M; \sigma m = m, \forall \sigma \in G\}.$$

Consider the functor $F(M) = M^G$ from the category of G-modules to the category of abelian groups.

(a) Show that the category of left G-modules is the same as the category of left modules over the ring $\mathbb{Z}[G]$. (Nothing fancy is warranted here; just describe the correspondence between the two categories.)

(b) Show that F is a left exact functor.

(c) Let t be a variable and let $G = \{t^n; n \in \mathbb{Z}\}$ be the infinite cyclic group generated by t. Let $N = \mathbb{Z}[G] = \mathbb{Z}[t, t^{-1}]$, and let M be the sub-G-module of N,

$$M = \{n \in N; n = n'(t-1) \text{ for some } n' \in N\} = \mathbb{Z}[t, t^{-1}](t-1).$$

Show that N and M are G-modules under left-multiplication. Show that as abelian groups $N/M \cong \mathbb{Z}$ and that the action of G on $\mathbb{Z}$, induced by this isomorphism, is trivial (i.e., $\sigma a = a$ for all $\sigma \in G, a \in \mathbb{Z}$).

(d) Use the exact sequence of G-modules

$$0 \longrightarrow M \longrightarrow N \longrightarrow \mathbb{Z} \longrightarrow 0$$

to show that F is not exact.

5. For $i \geq 0$, calculate $\operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ and $\operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Q})$.

6. Let B be an R-module. Show that the following are equivalent.

(i) B is projective.

(ii) For all R-modules C and $i \geq 1$, $\operatorname{Ext}^i_R(B, C) = 0$.

(iii) For all R-modules C, $\operatorname{Ext}^1_R(B, C) = 0$.

7. (will not be graded) Finish the proof of the snake lemma. Use the notation from lecture.