

HOMEWORK 1

DUE 14 APRIL 2017

1. Let R be a commutative ring (with $1 \neq 0$) and $\{M_i\}_{i \in I}$ be a collection of R -modules.
 - (a) Show that arbitrary direct sums and arbitrary direct products exist in the category of abelian groups.
 - (b) Show that $\bigoplus_{i \in I} M_i$ and $\prod_{i \in I} M_i$ as abelian groups inherit become R -module with $r \cdot (m_i)_{i \in I} = (r \cdot m_i)_{i \in I}$.
 - (c) Show that $\bigoplus_{i \in I} M_i$ is the direct sum in $R\text{-mod}$ of $\{M_i\}_{i \in I}$ and $\prod_{i \in I} M_i$ is the direct product in $R\text{-mod}$ of $\{M_i\}_{i \in I}$.
 - (d) Show that, for every R -module N ,

$$\text{Hom}_R \left(\bigoplus_{i \in I} M_i, N \right) \simeq \prod_{i \in I} \text{Hom}_R(M_i, N)$$

and

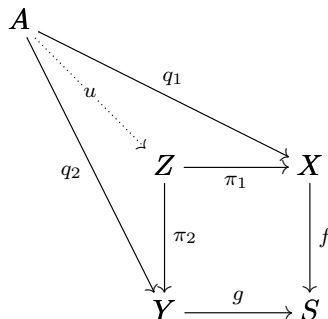
$$\text{Hom}_R \left(N, \prod_{i \in I} M_i \right) \simeq \prod_{i \in I} \text{Hom}_R(N, M_i).$$

- (e) Show that, for every R -module N ,

$$N \otimes_R \left(\bigoplus_{i \in I} M_i \right) \simeq \bigoplus_{i \in I} N \otimes_R M_i.$$

- (f) Does the tensor product also commute with direct products? Prove or give a counterexample.
 - (g) Is the tensor product of two free R -modules also free as an R -module? Prove or give a counterexample.
2. In this problem by “ring” we mean “ring with $1 \neq 0$ ” (but not necessarily commutative). Let \mathcal{R} be the category of rings. If R_1 and R_2 are rings, then let $R_1 \times R_2$ be their set-theoretic product, which can also be given the natural structure of a ring.
 - (a) Show that $R_1 \times R_2$ is the product of R_1 and R_2 in \mathcal{R} .
 - (b) Show that $R_1 \times R_2 \simeq R_1 \oplus R_2$ is not the coproduct of R_1 and R_2 in \mathcal{R} .
 Note: We’ll see later that coproducts do exist in the category $\mathcal{R}_{\text{comm}}$ of commutative rings; they’re called tensor products.

3. Let \mathcal{C} be a category. Let X, Y, S be objects in \mathcal{C} and $f : X \rightarrow S, g : Y \rightarrow S$ be morphisms in \mathcal{C} . A *fiber product* of f and g in \mathcal{C} (or by abuse of terminology, fiber product of X and Y over S) is an object Z in \mathcal{C} together with morphisms $\pi_1 : Z \rightarrow X$ and $\pi_2 : Z \rightarrow Y$ such that
 - (i) $g \circ \pi_2 = f \circ \pi_1$;
 - (ii) for any object A in \mathcal{C} and any morphisms $q_1 : A \rightarrow X, q_2 : A \rightarrow Y$ such that $g \circ q_2 = f \circ q_1$ there exists a unique morphism $u : A \rightarrow Z$ such that the diagram



is commutative.

Show that, if it exists, the fiber product Z of f and g is unique up to isomorphism. (The fiber product is denoted $X \times_S Y$.)

4. Let B be an abelian group. Let F_B be the functor from the category of abelian groups to itself defined for an abelian group A by

$$F_B(A) = \text{Hom}(B, A) = \{f : B \longrightarrow A; f \text{ is a group homomorphism}\}.$$

- (a) Show that F_B is a covariant functor.
- (b) Show that F_B is left exact.
- (c) Find a nontrivial abelian group B such that F_B is exact.
- (d) Is F_B always exact? Prove or find a counterexample.

5. Let G be a group. Denote $\mathbb{Z}[G]$ the free abelian group (or free \mathbb{Z} -module) on the set G . That is,

$$\mathbb{Z}[G] = \left\{ \sum_{\sigma \in G} a_\sigma \sigma; a_\sigma \in \mathbb{Z} \forall \sigma \in G \text{ and all but finitely many } a_\sigma \text{'s are equal to zero} \right\}$$

with the natural addition.

- (a) Show that $\mathbb{Z}[G]$ becomes a ring with the multiplication

$$\left(\sum_{\sigma \in G} a_\sigma \sigma \right) \cdot \left(\sum_{\tau \in G} b_\tau \tau \right) = \sum_{\sigma \in G} \left(\sum_{\sigma' \tau = \sigma} a_{\sigma'} b_\tau \right) \sigma.$$

(Do show that the multiplication is well-defined.)

- (b) What is the multiplicative identity element in this ring?
- (c) Show that the set R of finitely supported functions $f : G \longrightarrow \mathbb{Z}$ becomes a ring under the usual function addition and multiplication given by convolution. That is, the elements of R are maps of sets $f : G \longrightarrow \mathbb{Z}$ with the property that $f(\sigma) = 0$ for all but finitely many $\sigma \in G$; the addition is given by $(f + g)(\sigma) = f(\sigma) + g(\sigma)$ for all $\sigma \in G$; and the multiplication is given by

$$(f * g)(\sigma) = \sum_{\tau \in G} f(\tau)g(\tau^{-1}\sigma).$$

- (d) Show that $\mathbb{Z}[G]$ is naturally isomorphic to R (as rings).