HOMEWORK 1

DUE 14 APRIL 2017

1. Let R be a commutative ring (with $1 \neq 0$) and $\{M_i\}_{i\in I}$ be a collection of R-modules.

(a) Show that arbitrary direct sums and arbitrary direct products exist in the category of abelian groups.

(b) Show that $\bigoplus_{i\in I} M_i$ and $\prod_{i\in I} M_i$ as abelian groups inherit become R-module with $r\cdot(m_i)_{i\in I} = (r\cdot m_i)_{i\in I}$.

(c) Show that $\bigoplus_{i\in I} M_i$ is the direct sum in R-**mod** of $\{M_i\}_{i\in I}$ and $\prod_{i\in I} M_i$ is the direct product in R-**mod** of $\{M_i\}_{i\in I}$.

(d) Show that, for every R-module N,

$$\mathrm{Hom}_R\left(\bigoplus_{i\in I} M_i, N\right) \simeq \prod_{i\in I}\mathrm{Hom}_R(M_i, N)$$

and

$$\mathrm{Hom}_R\left(N, \prod_{i\in I} M_i\right) \simeq \prod_{i\in I}\mathrm{Hom}_R(N, M_i).$$

(e) Show that, for every R-module N,

$$N\otimes_R\left(\bigoplus_{i\in I} M_i\right) \simeq \bigoplus_{i\in I} N\otimes_R M_i.$$

(f) Does the tensor product also commute with direct products? Prove or give a counterexample.

(g) Is the tensor product of two free R-modules also free as an R-module? Prove or give a counterexample.

2. In this problem by "ring" we mean "ring with $1 \neq 0$" (but not necessarily commutative). Let $\mathcal{R}$ be the category of rings. If R_1 and R_2 are rings, then let $R_1\times R_2$ be their set-theoretic product, which can also be given the natural structure of a ring.

(a) Show that $R_1\times R_2$ is the product of R_1 and R_2 in $\mathcal{R}$.

(b) Show that $R_1\times R_2 \simeq R_1\oplus R_2$ is not the coproduct of R_1 and R_2 in $\mathcal{R}$.

Note: We'll see later that coproducts do exist in the category $\mathcal{R}_{\mathrm{comm}}$ of commutative rings; they're called tensor products.

3. Let $\mathcal{C}$ be a category. Let X, Y, S be objects in $\mathcal{C}$ and $f : X \longrightarrow S, g : Y \longrightarrow S$ be morphisms in $\mathcal{C}$. A *fiber product* of f and g in $\mathcal{C}$ (or by abuse of terminology, fiber product of X and Y over S) is an object Z in $\mathcal{C}$ together with morphisms $\pi_1 : Z\longrightarrow X$ and $\pi_2 : Z\longrightarrow Y$ such that

(i) $g\circ\pi_2 = f\circ\pi_1$;

(ii) for any object A in $\mathcal{C}$ and any morphisms $q_1 : A\longrightarrow X, q_2 : A\longrightarrow Y$ such that $g\circ q_2 = f\circ q_1$ there exists a unique morphism $u : A\longrightarrow Z$ such that the diagram

$$\begin{tikzcd}
A \arrow[drr, "q_1"] \arrow[ddr, "q_2"'] \arrow[dr, dotted, "u"] & & \\
& Z \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"] & S
\end{tikzcd}$$

is commutative.

Show that, if it exists, the fiber product Z of f and g is unique up to isomorphism. (The fiber product is denoted $X \times_S Y$.)

4. Let B be an abelian group. Let F_B be the functor from the category of abelian groups to itself defined for an abelian group A by

$$F_B(A) = \mathrm{Hom}(B, A) = \{f : B \longrightarrow A; f \text{ is a group homomorphism}\}.$$

(a) Show that F_B is a covariant functor.
(b) Show that F_B is left exact.
(c) Find a nontrivial abelian group B such that F_B is exact.
(d) Is F_B always exact? Prove or find a counterexample.

5. Let G be a group. Denote $\mathbb{Z}[G]$ the free abelian group (or free $\mathbb{Z}$-module) on the set G. That is,

$$\mathbb{Z}[G] = \left\{ \sum_{\sigma \in G} a_\sigma \sigma \, ; a_\sigma \in \mathbb{Z} \, \forall \sigma \in G \text{ and all but finitely many } a_\sigma\text{'s are equal to zero} \right\}$$

with the natural addition.

(a) Show that $\mathbb{Z}[G]$ becomes a ring with the multiplication

$$\left(\sum_{\sigma \in G} a_\sigma \sigma \right) \cdot \left(\sum_{\tau \in G} b_\tau \tau \right) = \sum_{\sigma \in G} \left(\sum_{\sigma' \tau = \sigma} a_{\sigma'} b_\tau \right) \sigma.$$

(Do show that the multiplication is well-defined.)

(b) What is the multiplicative identity element in this ring?
(c) Show that the set R of finitely supported functions $f : G \longrightarrow \mathbb{Z}$ becomes a ring under the usual function addition and multiplication given by convolution. That is, the elements of R are maps of sets $f : G \longrightarrow \mathbb{Z}$ with the property that $f(\sigma) = 0$ for all but finitely many $\sigma \in G$; the addition is given by $(f + g)(\sigma) = f(\sigma) + g(\sigma)$ for all $\sigma \in G$; and the multiplication is given by

$$(f * g)(\sigma) = \sum_{\tau \in G} f(\tau) g(\tau^{-1}\sigma).$$

(d) Show that $\mathbb{Z}[G]$ is naturally isomorphic to R (as rings).