HOMEWORK 6

DUE 13 MAY 2016

Part I.

1. Let R be a ring and $F : M \longrightarrow N$ a homomorphism of R-modules. Prove that the following are equivalent.

(a) f is surjective.

(b) $f_\mathfrak{p} : M_\mathfrak{p} \longrightarrow N_\mathfrak{p}$ is surjective for each prime ideal $\mathfrak{p}$ of R.

(c) $f_\mathfrak{m} : M_\mathfrak{m} \longrightarrow N_\mathfrak{m}$ is surjective for each maximal ideal $\mathfrak{m}$ of R.

2. Let p be a prime number. For $n \geq m$ let $f_{nm} : \mathbb{Z}/p^n\mathbb{Z} \longrightarrow \mathbb{Z}/p^m\mathbb{Z}$ be the canonical projection, i.e. $f_{nm}(a \bmod p^n) = a \bmod p^m$.

(a) Show that $\{\mathbb{Z}/p^n\mathbb{Z}\}$ with homomorphisms f_{nm} forms an inverse system of commutative rings. Let $\mathbb{Z}_p$ denote $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$

(b) Find the canonical image of $\mathbb{Z}$ in $\mathbb{Z}_p$ and show that $\mathbb{Z}_p$ is an integral domain.

(c) Show that $\mathbb{Z}_p$ is a local ring and an principal ideal domain.

The ring $\mathbb{Z}_p$ is called the ring of p-adic integers.

3. Let p be a prime and let R be the set of formal power series in p :

$$R = \left\{ \sum_{n=0}^{\infty} a_n p^n \,; a_n = 0, 1, \ldots, p-1 \right\}.$$

(a) Show that R is a commutative ring under the addition and multiplication of power series (do show that multiplication makes sense!).

(b) Show that $\mathbb{Z}_p$ is naturally isomorphic to R.

Bonus Let $\mathbb{N}$ be the set of positive integers ordered by divisibility. Observe that

$$\{\mathbb{Z}/n\mathbb{Z}\}_{n \in \mathbb{N}}$$

forms an inverse system of commutative rings with the canonical homomorphisms $\mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z}$ for $m \mid n$. Let $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$. Show that

$$\hat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p.$$

Part II. From Atiyah-Macdonald

Chapter 5: 1, 3, 4, 8, 9