# HOMEWORK 3

DUE 22 APRIL 2016

## Part I.

**1.** Let $R$ be a ring and $\{M_i\}_{i\in I}$ be a collection of $R$-modules.

(a) Show that direct sums and arbitrary direct products exist in the category of abelian groups.

(b) Show that $\bigoplus_{i\in I} M_i$ and $\prod_{i\in I} M_i$ as abelian groups inherit become $R$-module with $r\cdot(m_i)_{i\in I} = (r\cdot m_i)_{i\in I}$.

(c) Show that $\bigoplus_{i\in I} M_i$ is the direct sum in $R$-**mod** of $\{M_i\}_{i\in I}$ and $\prod_{i\in I} M_i$ is the direct product in $R$-**mod** of $\{M_i\}_{i\in I}$.

(d) Show that, for every $R$-module $N$,

$$\mathrm{Hom}_R\left(\bigoplus_{i\in I} M_i, N\right) \simeq \prod_{i\in I} \mathrm{Hom}_R(M_i, N)$$

and

$$\mathrm{Hom}_R\left(N, \prod_{i\in I} M_i\right) \simeq \prod_{i\in I} \mathrm{Hom}_R(N, M_i).$$

(e) Show that, for every $R$-module $N$,

$$N \otimes_R \left(\bigoplus_{i\in I} M_i\right) \simeq \bigoplus_{i\in I} N \otimes_R M_i.$$

(f) Does the tensor product also commute with direct products? Prove or give a counterexample.

(g) Is the tensor product of two free $R$-modules also free as an $R$-module? Prove or give a counterexample.

**2.** Write down explicitly the isomorphism $\mathrm{Hom}_R(M\otimes_R N, P)\longrightarrow \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P))$ and show that it is functorial, i.e. for each pair of $R$-module homomorphisms $f: M'\longrightarrow M$ and $g: P\longrightarrow P'$, and for any $R$-module $N$ the following diagram is commutative:

$$\begin{array}{ccc}
\mathrm{Hom}_R(M\otimes_R N, P) & \xrightarrow{\approx} & \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P)) \\
\downarrow{\scriptstyle g\circ(-)\circ(f\otimes 1_N)} & & \downarrow{\scriptstyle g_*\circ(-)\circ f} \\
\mathrm{Hom}_R(M'\otimes_R N, P') & \xrightarrow[\approx]{} & \mathrm{Hom}_R(M', \mathrm{Hom}_R(N,P'))
\end{array}$$

where $g_*$ is the pushforward of $g$.

**Part II.** From Atiyah-MacDonald

**Chapter 2:** 1, 2, 9, 11, 12

**Chapter 3:** 15