HOMEWORK 2

DUE 15 APRIL 2016

Part I.

1. Let R be a ring and M, N be R-modules.

(a) Write down the definitions of the functors from the category of R-modules to itself $\operatorname{Hom}_R(M, -) : R\textbf{-mod} \longrightarrow R\textbf{-mod}$ and $\operatorname{Hom}_R(-, N) : R\textbf{-mod} \longrightarrow R\textbf{-mod}$. Check that the two functors are well-defined. Which one is covariant and which one is contravariant?

(b) Show that in R**-mod** a sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z$$

is exact if and only if the sequence

$$0 \longrightarrow \operatorname{Hom}_R(M, X) \xrightarrow{f_*} \operatorname{Hom}_R(M, Y) \xrightarrow{g_*} \operatorname{Hom}_R(M, Z)$$

is exact for any R-module M. (The maps f_*, g_* are called the *pushforward* of f and g respectively.)

(c) Show that in R**-mod** a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is exact if and only if the sequence

$$0 \longrightarrow \operatorname{Hom}_R(Z, N) \xrightarrow{g^*} \operatorname{Hom}_R(Y, N) \xrightarrow{f^*} \operatorname{Hom}_R(X, N)$$

is exact for any R-module N. (The maps f^*, g^* are called the *pullback* of f and g respectively.)

(d) Prove that the functor $\operatorname{Hom}_R(M, -)$ is left exact. Prove the similar exactness result for $\operatorname{Hom}_R(-, N)$. (We call that *left-exactness* for *contravariant* functors.)

(e) Do the above results still hold if we think of the two functors as : $R\textbf{-mod} \longrightarrow \mathbb{Z}\textbf{-mod}$?

(f) Are the two functors exact? Prove or give counterexamples.

Part II. From Atiyah-MacDonald

Chapter 1: 2, 4, 15, 17

Bonus. From Atiyah-MacDonald

Chapter 1: 18, 21