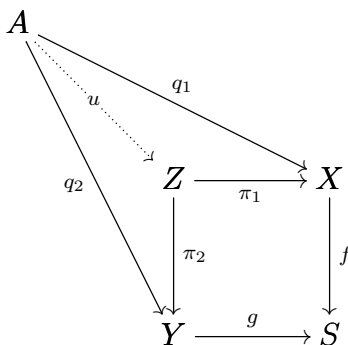


## HOMEWORK 1

DUE 8 APRIL 2016

1. Let  $\mathcal{C}$  be a category, and let  $U_1$  and  $U_2$  be objects in  $\mathcal{C}$ . Suppose  $U_1$  and  $U_2$  are both universally attracting. Show that there is a unique isomorphism  $i : U_1 \rightarrow U_2$ . (For future reference, the same is true if they're both universally repelling, with the same proof.)
2. Remember that by "ring" we mean "ring with 1." Let  $\mathcal{R}$  be the category of rings. If  $R_1$  and  $R_2$  are rings, then let  $R_1 \times R_2$  be their set-theoretic product, which can also be given the natural structure of a ring.
  - (a) Show that  $R_1 \times R_2$  is the product of  $R_1$  and  $R_2$  in  $\mathcal{R}$ .
  - (b) Show that  $R_1 \times R_2 \simeq R_1 \oplus R_2$  is not the coproduct of  $R_1$  and  $R_2$  in  $\mathcal{R}$ .
 Note: We'll see later that coproducts do exist in the category  $\mathcal{R}_{\text{comm}}$  of commutative rings; they're called tensor products.
3. Let  $\mathcal{C}$  be a category. Let  $X, Y, S$  be objects in  $\mathcal{C}$  and  $f : X \rightarrow S, g : Y \rightarrow S$  be morphisms in  $\mathcal{C}$ . A *fiber product* of  $f$  and  $g$  in  $\mathcal{C}$  (or by abuse of terminology, fiber product of  $X$  and  $Y$  over  $S$ ) is an object  $Z$  in  $\mathcal{C}$  together with morphisms  $\pi_1 : Z \rightarrow X$  and  $\pi_2 : Z \rightarrow Y$  such that
  - (i)  $g \circ \pi_2 = f \circ \pi_1$ ;
  - (ii) for any object  $A$  in  $\mathcal{C}$  and any morphisms  $q_1 : A \rightarrow X, q_2 : A \rightarrow Y$  such that  $g \circ q_2 = f \circ q_1$  there exists a unique morphism  $u : A \rightarrow Z$  such that the diagram



is commutative.

Show that, if it exists, the fiber product  $Z$  of  $f$  and  $g$  is unique up to isomorphism. (The fiber product is denoted  $X \times_S Y$ .)

4. Let  $B$  be an abelian group. Let  $F_B$  be the functor from the category of abelian groups to itself defined for an abelian group  $A$  by

$$F_B(A) = \text{Hom}(B, A) = \{f : B \longrightarrow A; f \text{ is a group homomorphism}\}.$$

- Show that  $F_B$  is a covariant functor.
- Show that  $F_B$  is left exact.
- Find a nontrivial abelian group  $B$  such that  $F_B$  is exact.
- Is  $F_B$  always exact? Prove or find a counterexample.

5. Let  $G$  be a group. Denote  $\mathbb{Z}[G]$  the free abelian group (or free  $\mathbb{Z}$ -module) on the set  $G$ . That is,

$$\mathbb{Z}[G] = \left\{ \sum_{\sigma \in G} a_\sigma \sigma; a_\sigma \in \mathbb{Z} \forall \sigma \in G \text{ and all but finitely many } a_\sigma \text{'s are equal to zero} \right\}$$

with the natural addition.

- Show that  $\mathbb{Z}[G]$  becomes a ring with the multiplication

$$\left( \sum_{\sigma \in G} a_\sigma \sigma \right) \cdot \left( \sum_{\tau \in G} b_\tau \tau \right) = \sum_{\sigma \in G} \left( \sum_{\sigma' \tau = \sigma} a_{\sigma'} b_\tau \right) \sigma.$$

(Do show that the multiplication is well-defined.)

- What is the multiplicative identity element in this ring?
- Show that the set  $R$  of finitely supported functions  $f : G \longrightarrow \mathbb{Z}$  becomes a ring under the usual function addition and multiplication given by convolution. That is, the elements of  $R$  are maps of sets  $f : G \longrightarrow \mathbb{Z}$  with the property that  $f(\sigma) = 0$  for all but finitely many  $\sigma \in G$ ; the addition is given by  $(f + g)(\sigma) = f(\sigma) + g(\sigma)$  for all  $\sigma \in G$ ; and the multiplication is given by

$$(f * g)(\sigma) = \sum_{\tau \in G} f(\tau) g(\tau^{-1} \sigma).$$

- Show that  $\mathbb{Z}[G]$  is naturally isomorphic to  $R$  (as rings).