1. Compute the double integral

$$\iint_D \frac{y}{x^2+y^2}\,dA$$

where D is the right half of the unit disk (i.e. the half of the unit disk $x^2+y^2 \leq 1$ where $x \geq 0$).

Solution. The domain D is described by the inequalities

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 1.$$

The function $f(x,y) = \frac{y}{x^2+y^2}$ in polar coordinates is

$$f(x,y) = \frac{y}{x^2+y^2} = \frac{r\sin\theta}{(r\cos\theta)^2+(r\sin\theta)^2} = \frac{r\sin\theta}{r^2(\cos^2\theta+\sin^2\theta)} = \frac{\sin\theta}{r}.$$

Using change of variables in polar coordinates gives

$$\begin{aligned}
\iint_D \frac{y}{x^2+y^2}\,dA &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \left(\frac{\sin\theta}{r}\right) r\,dr\,d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \sin\theta\,dr\,d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta \left(\int_0^1 1\,dr\right) d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\,d\theta \\
&= (-\cos\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= 0.
\end{aligned}$$

Alternate solution. The double integral is the signed volume of the region D between the graph of $f(x,y) = \frac{y}{x^2+y^2}$ and the xy-plane. Note that $f(x,y)$ takes opposite values at (x,y) and $(x,-y)$:

$$f(x,-y) = \frac{-y}{x^2+(-y)^2} = -\frac{y}{x^2+y^2} = -f(x,y).$$

By symmetry, the signed volume of the region below the xy-plane where $y \leq 0$ cancels with the signed volume of the region above the xy-plane where $y \geq 0$. Thus

$$\iint_D \frac{y}{x^2+y^2}\,dA = 0.$$

□