Name: ______________________

Section #: Section time:

1. (25 points) Calculate the work done by $\vec{F} = x(y^2+z^2)\hat{\imath} - z\hat{\jmath} + y\hat{\mathbf{k}}$ on the portion of the curve $x = t, y = \cos t, z = \sin t$ from $(0,1,0)$ to $(2\pi, 1, 0)$.

$$\text{work} = \int_C \vec{F}\cdot d\vec{r} = \int_C x(y^2+z^2)\,dx - z\,dy + y\,dz$$

on C: $t=0 \Rightarrow x=0,\ y=1,\ z=0$

$t=2\pi \Rightarrow x=2\pi,\ y=1,\ z=0$

C: $x=t \quad y=\cos t \quad z=\sin t$

$dx=dt \quad dy=-\sin t\,dt \quad dz=\cos t\,dt$

$$\text{work} = \int_0^{2\pi} t(\underbrace{\cos^2 t+\sin^2 t}_{=1})\,dt - \sin t(-\sin t\,dt) + \cos t(\cos t\,dt)$$

$$= \int_0^{2\pi} (t + \underbrace{\sin^2 t + \cos^2 t}_{=1})\,dt = \int_0^{2\pi} (t+1)\,dt =$$

$$= \frac{t^2}{2} + t \Big|_{t=0}^{t=2\pi} = \frac{4\pi^2}{2} + 2\pi = \boxed{2\pi^2 + 2\pi}$$

2. (a) (5 points) Give the formula of the scalar curl of a vector field $\vec{F} = M(x,y)\hat{\imath} + N(x,y)\hat{\jmath}$.

$$\text{curl } \vec{F} = N_x - M_y$$

(b) (5 points) Use the curl to show that $\vec{F} = (2x+5y^2)\hat{\imath} + (10xy - 6y^2)\hat{\jmath}$ is a gradient field.

$$\text{curl } \vec{F} = \frac{\partial}{\partial x}(10xy - 6y^2) - \frac{\partial}{\partial y}(2x+5y^2)$$

$$= 10y - 10y = 0$$

$\vec{F}$ defined and differentiable everywhere and curl $\vec{F} = 0$, so $\vec{F}$ is a gradient field

(c) (15 points) Find a potential function for $\vec{F}$.

potential $f(x,y)$ s.t. $\nabla f = \vec{F}$

So $f_y = 10xy - 6y^2$

$f_x = 2x + 5y^2 \Rightarrow f = x^2 + 5xy^2 + c(y)$

$f_y = 10xy + c'(y)$

$f_y = 10xy - 6y^2 \quad \Rightarrow$

$\Rightarrow c'(y) = -6y^2 \Rightarrow c(y) = -2y^3$ (+ constant)

$\therefore f = x^2 + 5xy^2 - 2y^3$ (+ constant)

Check: $f_x = 2x + 5y^2 \quad f_y = 10xy - 6y^2$

(d) (10 points) Find the work done by $\vec{F}$ along the curve $x = 2 + y^2(2-y), 0 \leq y \leq 1$.

work $= \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\text{end point}) - f(\text{start point})$

where $f(x,y)$ is from (c)

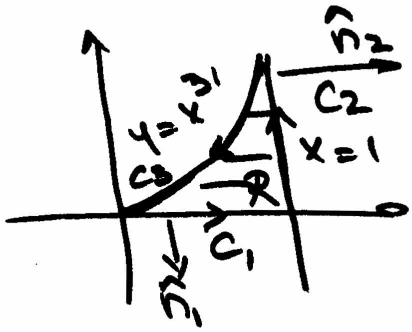
start point : $y=0$ $x=2$ $f(2,0) = 4$

end point : $y=1$ $x=3$ $f(3,1) = 9+15-2 = 22$

work $= f(3,1) - f(2,0) = 22-4 = \underline{\underline{18}}$

3. Consider the region R enclosed by the x-axis, $x = 1$ and $y = x^3$. *It is not unreasonable for flux to be negative.*

(a) (10 points) Use the normal form of Green's theorem to find the flux of $\vec{F} = (y^2 - 1)\hat{\jmath}$ out of R.



Flux out of $R = \int_C \vec{F} \cdot \hat{n}\, ds$

where $C = C_1 + C_2 + C_3$

Green's thm: $\int_C \vec{F} \cdot \hat{n}\, ds = \iint_R (\operatorname{div} \vec{F})\, dA$

$\operatorname{div} \vec{F} = \frac{\partial}{\partial y}(y^2 - 1) = 2y$ and in R: $0 \le x \le 1,\ 0 \le y \le x^3$

$$\text{flux} = \int_0^1 \int_0^{x^3} 2y\, dy\, dx = \int_0^1 y^2 \Big|_{y=0}^{y=x^3} dx = \int_0^1 x^6\, dx = \frac{1}{7}$$

(b) (20 points) Find the flux out of R through the horizontal segment C_1 and the vertical segment C_2.

on C_1: $\hat{n} = -\hat{\jmath} \Rightarrow \vec{F} \cdot \hat{n} = y^2 - 1 \underset{y=0.}{=} -1$

$0 \le x \le 1,\ y = 0$

So flux across $C_1 = \int_{C_1} \vec{F} \cdot \hat{n}\, ds = \int_0^1 -1\, dx = \underline{\underline{-1}}$

on C_2: $\hat{n} = \hat{\imath} \Rightarrow \vec{F} \cdot \hat{n} = 0 \Rightarrow$ flux across $C_2 = \int_{C_2} \vec{F} \cdot \hat{n}\, ds = \underline{\underline{0}}$

(c) (10 points) Use parts (a) and (b) to find the flux out of the third side C_3.

Flux out of C_3 = flux out of R − flux across C_1 − flux across C_2

$$= \int_C \vec{F}\cdot\hat{n}\,ds - \int_{C_1} \vec{F}\cdot\hat{n}\,ds - \int_{C_2} \vec{F}\cdot\hat{n}\,ds$$

$$= \frac{1}{7} - 1 - 0 = \boxed{-\frac{6}{7}}$$

Continuation of work on problem number ________
(Detach and recycle this page if it is not part of your solutions.)