# HOMEWORK 6

DUE 21 FEBRUARY 2014

**1.** Let $R$ be an Euclidean ring with structure map $\varphi$. Let $A \neq 0$ be an $m \times n$ matrix with coefficients in $R$. The purpose of this exercise is to show that there exist $P \in \mathrm{GL}(m, R)$ and $Q \in \mathrm{GL}(m, R)$ both products of elementary matrices such that

$$P^{-1}AQ = \begin{pmatrix} d_1 & & & & & & \\ & d_2 & & & & & \\ & & \ddots & & & & \\ & & & d_k & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} \tag{1}$$

with $d_1 \mid d_2 \mid \cdots \mid d_k$.

(a) Show that there exist matrices $P_1$ and $Q_1$ products of elementary matrices such that $A_1 = P_1^{-1}AQ_1$ has the property that $\varphi(a_{11}) \leq \varphi(a_{ij})$ for any $a_{ij} \neq 0$. (Nothing fancy here, just move the right element to the upper left corner).

(b) Show that there exist matrices $P_2$ and $Q_2$ products of elementary matrices such that $A_2 = P_2^{-1}AQ_2$ has the property from part (a) and $a_{1j} = 0$ for all $j \geq 2$. (You might have to go back to step (a) for this, but you should argue that you only have to do that finally many times.)

(c) Show that there exist matrices $P_3$ and $Q_3$ products of elementary matrices such that $A_3 = P_3^{-1}AQ_3$ has the property from part (b) and $a_{i1} = 0$ for all $i \geq 2$. (You might have to go back to step (a) or (b) for this, but you should argue that you only have to do that finally many times.)

(d) Show that there exist matrices $P_4$ and $Q_4$ products of elementary matrices such that $A_4 = P_4^{-1}AQ_4$ is of the form

$$A_4 = \begin{pmatrix} d_1 & 0 \dots 0 \\ 0 & \\ \vdots & M \\ 0 & \end{pmatrix}$$

where $d_1$ divides every entry of $M$. (You might have to go back to one of the previous steps for this, but you should argue that you only have to do that finally many times and each time things are improving.)

(e) Show that there exist matrices $P$ and $Q$ products of elementary matrices such that $P^{-1}AQ$ of the form (1). (Just apply the previous procedure to $M$ and repeat.)

**2.** Let $R = \mathbb{Z}[\sqrt{-5}]$. Find a presentation matrix for the ideal $(2, 1+\sqrt{-5})$ as an $R$-module.

**3.** Find a direct sum of cyclic groups isomorphic to the abelian group presented by the matrix

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

**4.** Write the abelian group generated by $x$ and $y$ with the relation $3x + 4y = 0$ as a direct sum of cyclic groups.

**5.** Let $T$ be the linear operator on $\mathbb{C}^2$ whose matrix is $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$. Is the corresponding $\mathbb{C}[t]$-module cyclic?

**6.** Let $V$ be an $F[t]$-module and let $\mathbf{B} = (v_1, \ldots, v_n)$ be a basis for $V$ as $F$-vector space. Let $B$ be the matrix of $T$ with respect to the basis $\mathbf{B}$. Prove that $A = tI_n - B$ is a presentation matrix for $V$ as an $F[t]$-module.

**7.** In how many ways can the additive group $\mathbb{Z}/5\mathbb{Z}$ be given a structure of $\mathbb{Z}[\sqrt{-1}]$-module?

From Dummit and Foote: section **12.2** problems 6, 7, 8; section **12.3** problems 5, 15, 24.