HOMEWORK 3

DUE 31 JANUARY 2014

1. Let G be a group. Denote $\mathbb{Z}[G]$ the free abelian group (or free $\mathbb{Z}$-module) on the set G. That is,

$$\mathbb{Z}[G] = \left\{ \sum_{\sigma \in G} a_\sigma \sigma \,;\, a_\sigma \in \mathbb{Z}\, \forall \sigma \in G \text{ and all but finitely many } a_\sigma\text{'s are equal to zero} \right\}$$

with the natural addition.

(a) Show that $\mathbb{Z}[G]$ becomes a ring with the multiplication

$$\left(\sum_{\sigma \in G} a_\sigma \sigma\right) \cdot \left(\sum_{\tau \in G} b_\tau \tau\right) = \sum_{\sigma \in G} \left(\sum_{\sigma' \tau = \sigma} a_{\sigma'} b_\tau\right) \sigma.$$

(Do show that the multiplication is well-defined.)

(b) What is the unit in this ring?

(c) Show that the set R of finitely supported functions $f : G \longrightarrow \mathbb{Z}$ becomes a ring under the usual function addition and multiplication given by convolution. That is, the elements of R are maps of sets $f : G \longrightarrow \mathbb{Z}$ with the property that $f(\sigma) = 0$ for all but finitely many $\sigma \in G$; the addition is given by $(f + g)(\sigma) = f(\sigma) + g(\sigma)$ for all $\sigma \in G$; and the multiplication is given by

$$(f * g)(\sigma) = \sum_{\tau \in G} f(\tau) g(\tau^{-1}\sigma).$$

(d) Show that $\mathbb{Z}[G]$ is naturally isomorphic to R (as rings).

2. Let G be a group. A (left) *G-module* is an abelian group M on which there is a G action which satisfies for all $m, m' \in M$ and $\sigma, \tau \in G$,

$$\begin{aligned} 1_G m &= m, \\ \sigma(\tau m) &= (\sigma\tau) m, \\ \sigma(m + m') &= \sigma m + \sigma m'. \end{aligned}$$

That is, there is a group homomorphism $G \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(M) : \sigma \mapsto \sigma(\cdot)$. A morphism of G-modules $f : M \longrightarrow N$ is a group homomorphism which also satisfies $f(\sigma m) = \sigma f(m)$, for $m \in M$ and $\sigma \in G$. For a G-module M, the subgroup of G-*invariant* elements of M is

$$M^G := \{m \in M; \sigma m = m, \forall \sigma \in G\}.$$

Let F be the functor from the category of G-modules to the category of abelian groups, defined by $F(M) = M^G$. Show

(a) Show that the category of left G-modules is the same as the category of left modules over the ring $\mathbb{Z}[G]$. (Nothing fancy is warranted here; just describe the correspondence between the two categories.)

(b) Show that F is a left exact functor.

(c) Let t be a variable and let $G = \{t^n; n \in \mathbb{Z}\}$ be the infinite cyclic group generated by t. Let $N = \mathbb{Z}[G] = \mathbb{Z}[t, t^{-1}]$, and let M be the sub-G-module of N,

$$M = \{n \in N; n = n'(t-1) \text{ for some } n' \in N\} = \mathbb{Z}[t, t^{-1}](t-1).$$

Show that N and M are G-modules under left-multiplication. Show that as abelian groups $N/M \cong \mathbb{Z}$ and that the action of G on $\mathbb{Z}$, induced by this isomorphism, is trivial (i.e., $\sigma a = a$ for all $\sigma \in G, a \in \mathbb{Z}$). Use the exact sequence of G-modules

$$0 \longrightarrow M \longrightarrow N \longrightarrow \mathbb{Z} \longrightarrow 0$$

to show that F is not exact.

3. Let R be a commutative ring and S a multiplicative subset of R. For any R-module M define $S^{-1}M$ is a manner similar to the one used to define $S^{-1}R$. Recall here that if R is not an integral domain, the equivalence relation that defines the localization is more complicated. Namely, $S^{-1}M$ is the set of equivalence classes of elements in $M \times S$ with respect to the equivalence relation

$$(x, s) \sim (x', s') \iff \exists s_1 \in S \text{ s.t. } s_1(s'x - sx') = 0.$$

(a) Show that $S^{-1}M$ is an $S^{-1}R$-module.

(b) Show that the localization functor $S^{-1}(\cdot) : R\text{-mod} \longrightarrow (S^{-1}R)\text{-mod}$ is an exact functor from the category of R-modules to the category of $S^{-1}R$-modules.

4. Let R be a commutative ring and $\mathfrak{p}$ a prime ideal of R. If $S = R \setminus \mathfrak{p}$, then the localization $S^{-1}M$ is denoted by $M_{\mathfrak{p}}$.

(a) Show that the natural map

$$M \longrightarrow \prod_{\mathfrak{m} \text{ maximal ideal of } R} M_{\mathfrak{m}}$$

is injective. (Hint: consider the annihilator ideal $\mathrm{Ann}_R(x) = \{r \in R; rx = 0\}$, but do show that it is an ideal.)

(b) Show that a sequence of R-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact if and only if the sequence

$$0 \longrightarrow M'_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow M''_{\mathfrak{p}} \longrightarrow 0$$

is exact for all prime ideals $\mathfrak{p}$ of R.

5. Let $R = \mathbb{Z}[\sqrt{-6}] = \{a + b\sqrt{-6}; a, b, \in \mathbb{Z}\}$. Let $\mathfrak{a} = (2, \sqrt{-6})$ be the ideal of R generated by 2 and $\sqrt{-6}$.

(a) Show that $\mathfrak{a}$ is not a free R-module.

(b) Show that $\mathfrak{a}$ is a projective R-module.