HOMEWORK 2

DUE 24 JANUARY 2014

1. Let $\mathcal{C}$ be a category, and let U_1 and U_2 be objects in $\mathcal{C}$. Suppose U_1 and U_2 are both universally attracting. Show that there is a unique isomorphism $i : U_1 \longrightarrow U_2$. (For future reference, the same is true if they're both universally repelling, with the same proof.)

2. Remember that by "ring" we mean "ring with 1." Let $\mathcal{R}$ be the category of rings. If R_1 and R_2 are rings, then let $R_1 \times R_2$ be their set-theoretic product, which can also be given the natural structure of a ring.

(a) Show that $R_1 \times R_2$ is the product of R_1 and R_2 in $\mathcal{R}$.

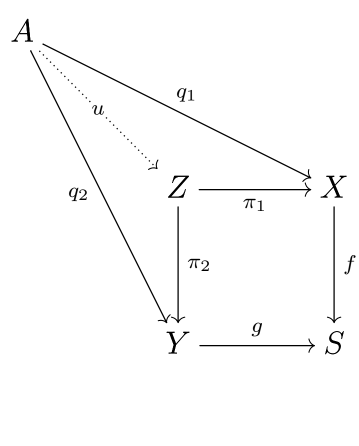
(b) Show that $R_1 \times R_2$ is not the coproduct of R_1 and R_2 in $\mathcal{R}$.

Note: We'll see later that coproducts do exist in the category $\mathcal{R}_{\text{comm}}$ of commutative rings; they're called tensor products.

3. Let $\mathcal{C}$ be a category. Let X, Y, S be objects in $\mathcal{C}$ and $f : X \longrightarrow S, g : Y \longrightarrow S$ be morphisms in $\mathcal{C}$. A *fiber product* of f and g in $\mathcal{C}$ (or by abuse of terminology, fiber product of X and Y over S) is an object Z in $\mathcal{C}$ together with morphisms $\pi_1 : Z \longrightarrow X$ and $\pi_2 : Z \longrightarrow Y$ such that

(i) $g \circ \pi_2 = f \circ \pi_1$;

(ii) for any object A in $\mathcal{C}$ and any morphisms $q_1 : A \longrightarrow X, q_2 : A \longrightarrow Y$ such that $g \circ q_2 = f \circ q_1$ there exists a unique morphism $u : A \longrightarrow Z$ such that the diagram



is commutative.

Show that, if it exists, the fiber product Z of f and g is unique up to isomorphism. (The fiber product is denoted $X \times_S Y$.)

4. From Dummit and Foote, Section 10.5: 27.

5. Show that fiber products exist in the category of R-modules. (Use Exercise 27.)

6. Let B be an abelian group. Let F_B be the functor from the category of abelian groups to itself defined for an abelian group A by

$$F_B(A) = \mathrm{Hom}(B, A) = \{f : B \longrightarrow A; f \text{ is a group homomorphism}\}.$$

(a) Show that F_B is a covariant functor.
(b) Show that F_B is left exact.
(c) Find a nontrivial abelian group B such that F_B is exact.
(d) Is F_B always exact? Prove or find a counterexample.