FINAL EXAM

21 MARCH 2014

You may quote results proved in class or in the textbook, but not if they trivialize the problem. Also, avoid quoting results proved only in homework exercises. All rings are unital.
Problem 1 is worth 20 points, problems 2–8 are worth 10 points each.

1. Prove or disprove (i.e. give a counterexample and prove it is a counterexample!) the following statements.

(a) Every projective module is flat.

(b) Every flat module is projective.

(c) Every free module is projective.

(d) Every projective module is free.

(e) Every injective module is flat.

(f) Every flat module is injective.

(g) Every projective module is injective.

(h) Every injective module is projective.

(i) Every abelian group is injective.

2. Let R be a commutative ring and $I \subset R$ an ideal. Show that for every R-module M, M/IM is an R/I-module and that M/IM is canonically isomorphic to $R/I \otimes_R M$ as R/I-modules.

3. Let T be the linear operator on $\mathbb{R}^2$ whose matrix is $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$. Find the primary decomposition of the corresponding $\mathbb{R}[t]$-module V. Is V cyclic?

4. In how many ways can the additive group $\mathbb{Z}/5\mathbb{Z}$ be given a structure of $\mathbb{Z}[\sqrt{-1}]$-module?

5. (a) Find the splitting field K of $X^3 - 2$ over $\mathbb{Q}$ and $\mathrm{Gal}(K/\mathbb{Q})$.
(b) Find all the subfields of K, and identify each of them as the fixed field of a subgroup of $\mathrm{Gal}(K/\mathbb{Q})$.

6. Let L/K be a Galois extension with $\mathrm{Gal}(L/K) \simeq \mathbb{Z}_2 \times \mathbb{Z}_{12}$. How many intermediate fields F are there with
(a) $[F : K] = 4$?
(b) $[F : K] = 9$?
(c) $\mathrm{Gal}(L/F) \simeq \mathbb{Z}_4$?

7. Factor the following polynomials into irreducible factors over the given field. You have to prove that your factorization is irreducible.
(a) $X^9 - X$ over $\mathbb{F}_3$;
(b) $X^9 - X$ over $\mathbb{F}_9$;
(c) $X^{27} - X$ over $\mathbb{F}_3$;
(d) $X^{27} - X$ over $\mathbb{F}_9$.

8. Find the minimal polynomial of the Frobenius automorphism $\phi(x) = x^q$ over a finite field $\mathbb{F}_{q^n}$ seen as an n-dimensional vector space over the field $\mathbb{F}_q$ of characteristic p.