HOMEWORK 9

DUE **FRIDAY** 15 MARCH 2013

Recall that if K is a number field (i.e. finite degree extension of $\mathbb{Q}$), then its ring of integers $\mathcal{O}_K$ is defined to be the integral closure of $\mathbb{Z}$ in K.

1. Find the ring of integers in $\mathbb{Q}(\sqrt[3]{2})$. Justify your answer.

2. Let $\sigma, \bar{\sigma} : \mathbb{Q}(\sqrt{-1}) \hookrightarrow \mathbb{C}$ be the two embeddings of $\mathbb{Q}(\sqrt{-1})$ into $\mathbb{C}$. Show that $\|x\|_1 = |\sigma(x)|^2$ and $\|x\|_2 = |\bar{\sigma}(x)|^2$ both define the same archimedean place of $\mathbb{Q}(\sqrt{-1})$.

3. Let p be a prime number. Find all the extensions of the p-adic valuation v_p on $\mathbb{Q}$ to $\mathbb{Q}(\sqrt{-1})$. *Hint: It will depend on* $p \pmod 4$.

4. Let ζ_5 denote a primitive fifth root of unity in $\mathbb{C}$. Set $K = \mathbb{Q}(\zeta_5)$.

(a) Find all the embeddings of K into $\mathbb{C}$.

(b) For each embedding $\sigma : K \hookrightarrow \mathbb{C}$ define $\|x\|_\sigma = |\sigma(x)|^2$. Show that this defines an archimedean generalized absolute value on K. How many distinct archimedean places of K do they represent?

5. Same problem for $K = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ where ζ_7 is a primitive seventh root of unity.

6. Let $K = \mathbb{Q}(\sqrt{-5})$. For any $\alpha_1, \ldots, \alpha_m \in K$, denote by

$$[\alpha_1, \ldots, \alpha_m] = \left\{ \sum_{i=1}^{m} a_i \alpha_i ; a_1, \ldots, a_m \in \mathbb{Z} \right\}$$

the $\mathbb{Z}$-submodule of K generated by $\alpha_1, \ldots, \alpha_m$. Then $\mathcal{O}_K = [1, \sqrt{-5}]$. Which, if any, of the following three $\mathbb{Z}$-modules are ideals?

- $[19 + 7\sqrt{-5}, 43 + 16\sqrt{-5}]$
- $[15 + 14\sqrt{-5}, 34 + 32\sqrt{-5}]$
- $[-31 + 11\sqrt{-5}, -71 + 25\sqrt{-5}]$

7. Let L/K be a degree n extension of number fields. Assume the ring of integers $\mathcal{O}_K$ is a principal ideal domain. Show that every fractional ideal of $\mathcal{O}_L$ is a free $\mathcal{O}_K$-module of rank n.

8. Let $p > 2$ be an odd prime and $n > 1$ an integer coprime to p. Let α be a root of the polynomial $X^n - p \in \mathbb{Q}_p[X]$. Let $K = \mathbb{Q}_p(\alpha)$. We extend the p-adic valuation v_p to K as follows.

(a) Find an extension of $|\cdot|_p$ to K. Its equivalence class is the unique place of K above $|\cdot|_p$.

(b) Find the discrete valuation w on K associated to this place and an uniformizer $\pi \in K$.

(c) Find the group homomorphism $f : \mathbb{Z} = \mathrm{Im}(v_p) \to \mathbb{Z} = \mathrm{Im}(w)$ induced by the natural embedding of $\mathbb{Q}_p$ in K via the commutative diagram

$$\begin{array}{ccc} \mathbb{Q}_p & \hookrightarrow & K \\ v_p \downarrow & & \downarrow w \\ \mathbb{Z} & \xrightarrow{f} & \mathbb{Z} \end{array}$$

(d) Find $\ker(f)$ and $\mathrm{coker}(f)$. (This is the sense is which we think of the quotient group $\mathrm{Im}(w)/\mathrm{Im}(v_p)$.)

If you're curious, try to see what happens when you take $n = p^k$. But this is not a required part of this assignment,

9. Prove the *Weak approximation theorem*:

Let $|\cdot|_n$ with $1 \leq n \leq N$ be nontrivial non-equivalent generalized absolute values on a field F. For each n denote by F_n the topological space induced by $|\cdot|_n$ on F. Then the image Δ of F in the product topological space

$$X = \prod_{n=1}^{N} F_n$$

is dense in X. In other words, given $\alpha_n \in F, 1 \leq n \leq N$ and $\varepsilon > 0$ there exist $a \in F$ such that

$$|a - \alpha_n|_n < \varepsilon \text{ for all } 1 \leq n \leq N.$$

Hint: see Section 6, Chapter II in Cassels and Fröhlich.