HOMEWORK 8

DUE 6 MARCH 2013

1. Let R be a Dedekind domain, and denote by K its quotient field. Let M, N by two finitely generated R-modules that span the K-vector space V of $\dim_K V = n$. Show that $M \simeq N$ if and only if $[M : N]$ is a principal fractional ideal.

2. Show that any two norms on a finite dimensional real vector space are equivalent.

3. Now show that the same is true if you replace $\mathbb{R}$ by a field F endowed with a generalized absolute value $|\cdot|$ with the property that F is complete in the induced topology.

4. Formulate and prove Hensel's Lemma for a field K that is complete with respect to a discrete valuation v and its valuation ring $R = R_v$. This goes for all three versions that we stated in class. Consider each version a separate problem.

5. Let K be s field and A, B two rings that contain K. Let $C = A \otimes_K B$.
(a) Show that C is a K-algebra.
(b) If $\dim_K B = n < \infty$ and A is a topological ring (i.e. it has a topology with respect to which addition and multiplication are continuous maps), prove that C there is a $1-1$ correspondence between C and $\oplus_{i=1}^n A$.
(c) Give C the topology induced by the product topology on $\oplus_{i=1}^n A$ via the correspondence from (b). Show that the topology on C is independent of the specific correspondence.
(d) Show that in the same topology the addition and multiplication on C are continuous, i.e. that C is a topological ring.

This topology will be called the *tensor topology* on C.

6. Find the primes p for which -1 has a square root in $\mathbb{Q}_p$.