HOMEWORK 7

DUE 27 FEBRUARY 2013

1. Let R be a commutative ring. Show that the set $GL(n, R)$ of $n \times n$ invertible matrices with coefficients in R consists exactly of the $n \times n$ matrices with determinant a unit of R. Show that this set is a group under matrix multiplication.

2. Assume that R is an integral domain and M a free R-module of rank n. Fix a basis $x_1, \ldots, x_n$ of M over R. Show that for every R-linear automorphism ℓ of M there exists a matrix $A \in GL(n, R)$ such that

$$\ell(x) = A \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

where $x = \sum a_i x_i$. Is A unique?

3. Let $D \in \mathbb{Z}$ be a nonzero square-free integer.

(a) Show that an element $\alpha \in \mathbb{Q}(\sqrt{D})$ is integral over $\mathbb{Z}$ if and only if its minimal polynomial over $\mathbb{Q}$ has integer coefficients.

(b) Show that

$$\{\alpha \in \mathbb{Q}(\sqrt{D}); \alpha \text{ integral over } \mathbb{Z}\} = \begin{cases} \mathbb{Z}[\sqrt{D}] & D \equiv 2, 3 \pmod 4 \\ \mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right] & D \equiv 1 \pmod 4. \end{cases}$$

(c) Show that $\mathbb{Z}[\sqrt{D}]$ is integrally closed when $D \equiv 2, 3 \pmod 4$.

(d) If $D \equiv 1 \pmod 4$, find an element in $\mathbb{Q}(\sqrt{D}) \setminus \mathbb{Z}[\sqrt{D}]$ that is integral over $\mathbb{Z}[\sqrt{D}]$. Conclude that $\mathbb{Z}[\sqrt{D}]$ is not integrally closed.

Give examples of the following structures or prove that they don't exist. Justify your answers. Once again, Google is your friend.

4. An integral domain that is not a UFD.

5. A UFD that is not a PID. Exhibit an ideal that is not principal and prove that it is not principal.

6. A PID that is not an Euclidean ring.

7. A PID that is not a Dedekind domain.

8. An integral domain and a prime element in it that is not irreducible.

9. An integral domain and an irreducible element in it that is not prime.

10. A prime ideal in $\mathbb{Z}[\sqrt{-5}]$ that is not principal.

11. A prime ideal in $\mathbb{Z}[\sqrt{-3}]$ that is not principal.

12. A ring R and a finitely generated R-module M that is torsion-free but not free.

13. An integral domain R and a finitely generated R-module M that is torsion-free but not free.

14. A principal ideal domain R and a finitely generated R-module M that is torsion-free but not free.