HOMEWORK 6

DUE 20 FEBRUARY 2013

1. Show that a closed subset of a complete metric space is complete.

2. Let R be a commutative ring and $\mathfrak{p}$ a prime ideal of R. Assume that I, J are ideals of R and $IJ \subset \mathfrak{p}$. Show that $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$.

3. Show that the ring $\mathbb{Z}$ is integrally closed.

4. (a) Let $a + b\sqrt{-1} \in \mathbb{Z}[\sqrt{-1}]$. Find its minimal polynomial over $\mathbb{Q}$.
(b) Show that if an element $x \in \mathbb{Q}(\sqrt{-1})$ is integral over $\mathbb{Z}$ then its minimal polynomial has coefficients in $\mathbb{Z}$.
(c) Show that $\mathbb{Z}[\sqrt{-1}]$ is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{-1})$.

5. (a) Let $x + y\sqrt{-3} \in \mathbb{Q}(\sqrt{-3})$. Find its minimal polynomial over $\mathbb{Q}$.
(b) Show that if an element $x \in \mathbb{Q}(\sqrt{-3})$ is integral over $\mathbb{Z}$ then its minimal polynomial has coefficients in $\mathbb{Z}$.
(c) Show that $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{-3})$.

6. Let $A \subset B$ be commutative rings such that B is a finitely generated A-module. Show that any finitely generated B-module M is finitely generated as an A-module.

7. Let K be a number field and $\mathcal{O}_K$ its ring of integers. Show that K is the fraction field of the integral domain $\mathcal{O}_K$.

8. Let $A \subset B \subset C$ be commutative rings and denote by $\bar{A}$ the integral closure of A in B.
(a) Show that $\bar{A}$ is a ring.
(b) Show that the integral closure of A in C is the same as the integral closure of $\bar{A}$ in C.
(c) If B is integral over A and C is integral over B, show that C is integral over A.

9. Let A be an integral domain and K its quotient field. Let L/K be a finite field extension and B the integral closure of A in L.
(a) Prove that B is a ring.
(b) Show that L is the quotient field of B.

Hint: Lemma 5.21 from the notes will help with many of these exercises. So might Google and Atiyah & MacDonald.