HOMEWORK 5

DUE 13 FEBRUARY 2013

1. Let R be a commutative ring. Assume that I is an ideal of R and $R \setminus I = R^\times$. Show that R is a local ring with maximal ideal I.

2. Let R be a commutative ring and $\mathfrak{m}$ a maximal ideal of R. Show that $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an $R/\mathfrak{m}$-vector space for all $n \geq 0$.

3. Let F be a field and $K = F(X)$. Show that $v_\infty(f/g) = \deg g - \deg f$ is a discrete valuation on K and compute its valuation ring and residue field.

4. Prove the following version of Ostrowski's Theorem for $F(X)$.
Every non-trivial valuation on $K = F(X)$ that is trivial on F is equivalent either to $|\cdot|_f$ for some irreducible polynomial $f \in F[X]$ or to $|\cdot|_\infty$.
Here $|h|_f = \rho^{v_f(h)}$ and $|h|_\infty = \rho^{v_\infty(h)}$ for some $0 < \rho < 1$. In particular, all these generalized absolute values are non-archimedean.

5. Prove that if $F = \mathbb{F}_q$ is a finite field, the only non-trivial generalized absolute values on $F(X)$ are equivalent either to some $|\cdot|_f$ for some irreducible polynomial $f \in F[t]$ or to $|\cdot|_\infty$.

6. Let $F((t)) = \left\{ \sum_{n=-m}^{\infty} a_n t^n ; m \in \mathbb{Z}, a_n \in F \right\}$ be the field of Laurent series over the field F.
First show that $F((t))$ is indeed a field. Then show that

$$v\left(\sum_{n=-m}^{\infty} a_n t^n \right) = \inf\{n; a_n \neq 0\}$$

is a discrete valuation on $F((t))$ and compute its valuation ring.

7. With the same notation as above, show that $F((t))$ is complete with respect to the absolute value induced by the discrete valuation v.

8. Show that $\mathbb{Z}_p/p\mathbb{Z}_p \simeq \mathbb{Z}/p\mathbb{Z}$.

9. If v is a discrete valuation on the field K, show that $\mathfrak{p}_v^n, n \in \mathbb{Z}$ form a fundamental system of neighborhoods for 0 in the induced topology.