HOMEWORK 4

DUE 6 FEBRUARY 2013

Inverse limits

1. Let R be a commutative ring. An *inverse (projective) system* of R-modules is a sequence $M_n, n \geq 1$ of R-modules together with R-module homomorphisms $f_n : M_{n+1} \to M_n$. Define

$$M = \left\{ (x_n)_n \in \prod_{n=1}^{\infty} M_n; f_n(x_{n+1}) = x_n \right\}.$$

(a) Show that M is an R-module and that the map $p_n : M \to M_n, p_n((x_j)_j) = x_n$ is a homomorphism of R-modules for all $n \geq 1$. Show also that $f_n \circ p_{n+1} = p_n$ for all $n \geq 1$.

(b) Show that M has the following universality property. If N is an R-module and $g_n : N \to M_n, n \geq 1$ are R-linear maps such that $f_n \circ g_{n+1} = g_n$ for all $n \geq 1$, then there exists a *unique* R-linear map $g : N \to M$ such that $g_n = p_n \circ g$ for all $n \geq 1$.

(c) Show that M is the unique (up to isomorphism) R-module that satisfies (b).

An R-module M that satisfies the universality property (b) is called the *inverse (projective) limit* of the inverse system $(M_n, f_n)_n$ and we write

$$M = \varprojlim M_n.$$

2. (a) Show that $\mathbb{Z}/p^n\mathbb{Z}$ form an inverse system of abelian groups with the maps $f_n : \mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}, x + p^{n+1}\mathbb{Z} \mapsto x + p^n\mathbb{Z}$.

(b) Show that $\mathbb{Z}_p \simeq \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ with the natural homomorphisms given by $\varepsilon_n : \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$, $\varepsilon_n(x) = x \pmod{p^n}$.

3. Recall that $U_n = 1 + p^n\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ for all $n \geq 1$.

(a) Show that $A_n = U_1/U_n$ form an inverse system of abelian groups with $g_n : A_{n+1} \to A_n$, $g_n(xU_{n+1}) = xU_n$.

(b) Show that $U_1 \simeq \varprojlim U_1/U_n$ with the natural projections $p_n : U_1 \to U_1/U_n$.

4. Assume $n \geq 1$ and $p \neq 2$ or $n \geq 2$ and $p = 2$. Let $x \in U_n \setminus U_{n+1}$. Then $x^p \in U_{n+1} \setminus U_{n+2}$.

5. Assume $p \neq 2$.

(a) Choose $\alpha \in U_1 \setminus U_2$. Show that U_1/U_n is a cyclic group of order p^{n-1} generated by $\alpha_n = p_n(\alpha)$.

(b) Show that $\theta_n : \mathbb{Z}/p^{n-1}\mathbb{Z} \to U_1/U_n$, $\theta_n(\bar{x}) = \alpha_n^x$ is an isomorphism for all $n \geq 2$.

(c) Show that $g_n \circ \theta_{n+1} = \theta_n \circ f_{n-1}$.

(d) Show that $(\theta_n)_n$ induce an isomorphism $\theta : \mathbb{Z}_p \to U_1$.

(e) Compute $\theta\left(\sum_{r=0}^{\infty} b_r p^r\right)$.

6. Assume $p = 2$.

(a) Show that $1 + 2^2\mathbb{Z}_2 \simeq \mathbb{Z}_2$.

(b) Show that $1 + 2\mathbb{Z}_2 = \{\pm 1\} \times (1 + 2^2\mathbb{Z}_2)$.