HOMEWORK 3

DUE 30 JANUARY 2013

Fix a prime p. The first three exercises below concern the Teichmüller representatives of p-adic numbers.

1. Prove the $\mathbb{Q}_p$ always contains p solutions $a_0, \ldots, a_{p-1}$ to the equation

$$x^p - x = 0$$

with $a_j \equiv j \pmod{p}$. These p numbers are called the *Teichmüller representatives* of the numbers $\{0, 1, \ldots, p-1\}$ and can be used as a set of p-adic digits instead of the choice $0, 1, \ldots, p-1$ we made in class. That is, every p-adic number $x \in \mathbb{Q}_p$ has a unique Teichmüller representation

$$x = \sum_{r \geq -m} b_r p^r \text{ with } b_r \in \{a_0, \ldots, a_{p-1}\}.$$

2. Let $\alpha \in \mathbb{Z}_p$.

(a) Prove that

$$\alpha^{p^n} \equiv \alpha^{p^{n-1}} \pmod{p^n} \text{ for } n \geq 1.$$

(b) Prove that the sequence $\left(\alpha^{p^n}\right)_{n \geq 1}$ is convergent in $\mathbb{Q}_p$ and that its limit is the Teichmüller representative congruent to $\alpha \pmod{p}$.

3. Let $\mathrm{pr} : \mathbb{Z}_p \to \mathbb{F}_p$ be the natural projection, i.e. $\mathrm{pr}(a) = a \pmod{p}$. With the notation from the Problem 1, show that the map $\mathbb{F}_p \to \mathbb{Z}_p, j \mapsto a_j$ is the unique multiplicative map $f : \mathbb{F}_p \to \mathbb{Z}_p$ that has the property that $f \circ \mathrm{pr} = \mathrm{Id}_{\mathbb{F}_p}$.

4. (Eisenstein criterion) Let $F(X) = c_0 + c_1 X + \cdots + c_n X^n \in \mathbb{Z}_p[X]$ and assume that

$$c_j \equiv 0 \pmod{p}, 0 \leq j \leq n-1, \; c_n \not\equiv 0 \pmod{p}, \text{ and } c_0 \not\equiv 0 \pmod{p^2}.$$

Prove that $F(X)$ is irreducible in $\mathbb{Q}_p[X]$.

5. Show that for every positive integer n

$$v_p(n!) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor < \frac{n}{p-1}.$$

In particular,

$$|n!|_p > p^{-\frac{n}{p-1}}.$$

6. Show that every p-adic number that is sufficiently close to 1 is a pth power. That is, find an absolute constant $m \in \mathbb{Z}_{>0}$ (that does not depend on p) such that

$$x \in \mathbb{Q}_p, |x-1|_p < p^{-m} \implies \exists\, y \in \mathbb{Q}_p \text{ s.t. } x = y^p.$$

Note that your constant must work for all primes, including $p = 2$. What is the optimal such m? *Hint: Use the generalized binomial expansion.*

7. A *fundamental system of neighborhoods (neighborhood basis)* of a point x in a topological space X is a family $\mathcal{B}$ of neighborhoods of x such that for every neighborhood $U \subseteq X$ of x there exists $V \in \mathcal{B}$ such that $V \subseteq U$. Note that if one wants to prove continuity at a point, it is enough to do so for a neighborhood basis (assuming one can find a basis, of course).

Set

$$B = \bigcap_{m \geq 1,\, p \nmid m} \{x^m; x \in \mathbb{Q}_p^\times\}$$

the set of elements of $\mathbb{Q}_p^\times$ that are mth powers for every m coprime to p. Prove that the sets

$$A_n = \{x^{p^n}; x \in \mathbb{Q}_p^\times\} \cap B, n \geq 0$$

form a neighborhood basis of 1 in $\mathbb{Q}_p$. Prove that $a - 1 + A_n$ form a neighborhood basis of $a \in \mathbb{Q}_p$. *Hint: the A_n's are defined via algebraic means in light of the next problem, but it might help to consider the sets $U_n = 1 + p^n\mathbb{Z}_p$, especially U_1.*

8. Prove that $\mathbb{Q}_p$ has no nontrivial field automorphisms. Note that we are **not** assuming that the automorphisms are continuous. *Hint: you can use the previous problem to show that a field automorphism of $\mathbb{Q}_p$ is forced to be continuous.*