HOMEWORK 2

DUE **FRIDAY** 25 JANUARY 2013

1. Assume F is a field and $|\cdot|$ is a *generalized* absolute value on F. Prove that F can be embedded in a complete field $\hat{F}$ with a generalized absolute value $|\cdot|'$ extending $|\cdot|$ in such a way that $\hat{F}$ is the closure of F with respect to $|\cdot|'$. Further, $\hat{F}$ is unique up to isomorphism. (Hint: use the fact that any generalized absolute value is equivalent to an absolute value.)

2. Show that for $x \in \mathbb{Q}^\times$

$$\prod_{p \text{ prime or } p=\infty} |x|_p = 1.$$

3. Prove that if $x \in \mathbb{Q}$ and $|x|_p \leq 1$ for every prime p, then x is an integer.

4. Let p be a prime. Prove that any sequence of integers has a subsequence which is Cauchy with respect to $|\cdot|_p$.

5. Assume $a \in \mathbb{Q}_p$ has p-adic expansion $\sum_{n \geq -m} a_n p^n$. Find the p-adic expansion of $-a$.

6. Find the p-adic expansion of

(a) $(3 + 2 \times 7 + 0 \times 7^2 + 1 \times 7^3 + \dots)(2 + 6 \times 7 + 0 \times 7^2 + 1 \times 7^3 + \dots)$ in $\mathbb{Q}_7$ to four digits;

(b) $\dfrac{1}{3 + 1 \times 5 + 2 \times 5^2 + 2 \times 5^3 + \dots}$ in $\mathbb{Q}_5$ to four digits;

(c) $9 \times 11^2 - (2 \times 11^{-3} + 0 \times 11^{-2} + 3 + 7 \times 11^4 + \dots)$ in $\mathbb{Q}_{11}$ to four digits;

(d) $2/3$ in $\mathbb{Q}_2$;

(e) $-1/6$ in $\mathbb{Q}_7$;

(f) $-9/16$ in $\mathbb{Q}_5$.

7. Prove that the p-adic expansion of some $a \in \mathbb{Q}_p^\times$ terminates (i.e. there exists N such that $a_i = 0$ for all $i \geq N$) if and only if a is a positive rational number whose denominator is a power of p.

8. Prove that the p-adic expansion of some $a \in \mathbb{Q}_p^\times$ has repeating digits from some point on (i.e. there exists N and r such that $a_{i+r} = a_i$ for all $i \geq N$) if and only if $a \in \mathbb{Q}^\times$.

9. Prove that the infinite sum

$$1 + p + p^2 + \dots$$

converges to $\frac{1}{1-p}$ in $\mathbb{Q}_p$. What about

$$\sum_{n \geq 0} (-1)^n p^n$$

and

$$1 + (p-1)p + p^2 + (p-1)p^3 + p^4 + (p-1)p^5 + \dots?$$