HOMEWORK 1

DUE 16 JANUARY 2013

Fix a prime p. For any nonzero integer n set $v_p(n)$ to be the highest power of p that divides n. That is $v_p(n) = a$ such that $p^a \mid n$, but $p^{a+1} \nmid n$. We will also make the convention that $v_p(0) = \infty$.

1. Show that
$$v_p\left(\frac{m}{n}\right) = v_p(m) - v_p(n)$$
only depends on the fraction $\frac{m}{n}$ and not on the particular m and n. This means that v_p is a well-defined map $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$.

2. Show that for any $x, y \in \mathbb{Q}$
(a) $v_p(xy) = v_p(x) + v_p(y)$
(b) $v_p(x) \in \mathbb{Z}$ for any nonzero rational x;
(c) $v_p(x + y) = \min\{v_p(x), v_p(y)\}$ if $v_p(x) \neq v_p(y)$
(d) $v_p(x + y) \geq \min\{v_p(x), v_p(y)\}$

3. Show that $v_p : \mathbb{Q}^\times \to \mathbb{Z}$ is a surjective group homomorphism.

Define the p-adic absolute value on $\mathbb{Q}$ by
$$|x|_p = \begin{cases} \dfrac{1}{p^{v_p(x)}} & \text{if } x \neq 0 \\ \\ 0 & \text{if } x = 0. \end{cases}$$

4. Show that $|\cdot|_p$ is an absolute value on the field of rational numbers $\mathbb{Q}$.

5. Show that for any $x, y \in \mathbb{Q}$
(a) $|x + y|_p \leq \max\{|x|_p, |y|_p\}$
(b) $|x + y|_p = \max\{|x|_p, |y|_p\}$ if $|x|_p \neq |y|_p$

6. Show that if $q \neq p$ are two different primes, then $|\cdot|_p \not\sim |\cdot|_q$ on $\mathbb{Q}$.

7. Show that, if $a > 0$, then $|\cdot|_\infty^a$ is an absolute value if and only if $a \leq 1$. Here $|\cdot|_\infty$ denotes the usual absolute value.

8. Show that if $|\cdot|$ is a generalized absolute value on a field F, then addition, multiplication and taking inverses are continuous functions in the induced topology.

9. Compute the p-adic distance $|a - b|_p$ for
(a) $a = 7, b = 4, p = 3$;
(b) $a = 7, b = -20, p = 3$;
(c) $a = 1/9, b = 1/16, p = 5$;
(d) $a = 9!, b = 0, p = 3$.