HOMEWORK 4

DUE 7 FEBRUARY 2013

1. This exercise concerns the homomorphism $\chi_D : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}$ of Theorem 5 from the Jacobi Symbol notes posted on the website. Recall that $D \equiv 0, 1 \pmod 4$ and that $\chi_D([a]) = \left(\frac{D}{m}\right)$ where m is an odd positive integer with $m \equiv a \pmod D$.

If $D \equiv 1 \pmod 4$, show that

$$\chi_D([2]) = \begin{cases} 1 & \text{if } D \equiv 1 \pmod 8; \\ -1 & \text{if } D \equiv 5 \pmod 8). \end{cases}$$

2. Find all positive integer solutions to the equation

$$x^2 + 2y^2 = z^2.$$

3. These are two identities used by Euler.

(a) Prove that

$$(x^2 + ny^2)(s^2 + nt^2) = (sx \pm nty)^2 + n(tx \mp sy)^2.$$

(b) Generalize the above to find an identity of the form

$$(ax^2 + cy^2)(as^2 + ct^2) = (?)^2 + ac(?)^2.$$

4. (a) Show that if $x^5 + y^5 + z^5 = 0$, then

$$2(x+y+z)^5 = 5(x+y)(x+z)(y+z)\left[(x+y+z)^2 + x^2 + y^2 + z^2\right]$$

Use this to show that 5 divides one of the numbers x, y, z.

(b) Show that Fermat's equation

$$x^5 + y^5 = w^5$$

has no solution when 5 *does not* divide any of the numbers x, y, w.