MATH 100C – Final Exam Study Guide

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First, let me warn you that this is by no means a complete list of problems, or topics. Just highlights. The first thing you should do when preparing for the exam is to go through your notes, the relevant sections of the book and the homework problems. If you still have trouble with some of the topics encountered so far, take the book (or another abstract algebra book) and solve more problems related to that topic until you *really* understand how and why things work. The final exam covers everything we discussed throughout the term.

Topics

group actions, transitivity
Sylow subgroups and Sylow's theorems; applications
solvable groups
finite fields
field extensions, splitting fields, separability and normal extensions in towers
the Galois group of a field extension, its action on roots of polynomials
fixed subfields
Artin's Lemma
finite Galois extensions, intermediate extensions, the main theorem of Galois theory solvability by radicals,
cubic, quartic and quintic field extensions
cyclotomic polynomials
discriminants and the Galois group viewed as a group of permutations

Attention: always make sure that you know if a polynomial you're working with is irreducible or not!

Practice problems

This is a collection of extra problems to help you prepare for the final. Once again, the list is not complete and these problems do **not** imply anything about the content of the exam.

Notation: $\zeta_n = e^{2\pi i/n} \in \mathbb{C}$ and $\mathbb{F}_q$ is the finite field with q elements.

1. Go over HW.
2. Compute the discriminant of a quadratic polynomial $f = ax^2 + bx + c \in K[x]$.
3. Let $f = x^3 - s_1x^2 + s_2x - s_3 \in K[x]$ with $\operatorname{char} K \neq 3$.
   (a) Prove that the *Tschirnhausen transformation*
   $$y = x + \frac{s_1}{3}$$
   does not change the discriminant of f.
   (b) Determine the coefficients of the polynomial obtained from f via the Tschirnhausen transformation.

4. (a) Prove that the discriminant of a real cubic is non-negative if and only if the cubic has 3 real roots.
   (b) Suppose that a real quartic polynomial has a positive discriminant. What can you say about the number of real roots?

5. Let $f \in K[x]$ be a polynomial of degree n and F its splitting field over K. Prove that $[F : K]$ divides $n!$.

6. Let $K = \mathbb{F}_2(t)$ be the field of rational functions of one variable with coefficients in $\mathbb{F}_2$. Prove that the polynomial $x^2 - t$ is irreducible over K and that it has a double root in a splitting field.

7. Let α be a root of the polynomial $f(x) = x^3 + x + 1 \in \mathbb{Q}[x]$. Let K be the splitting field of $f(x)$ over $\mathbb{Q}$. Is $\sqrt{-31}$ in $\mathbb{Q}(\alpha)$? Is it in K?

8. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$.
   (a) Prove that $K/\mathbb{Q}$ is a Galois extension and determine its Galois group.
   (b) Find all the intermediate fields F in $K/\mathbb{Q}$. Which ones are Galois extensions of $\mathbb{Q}$?

9. Let L/K be a Galois extension with $\mathrm{Gal}(L/K) \simeq \mathbb{Z}_2 \times \mathbb{Z}_{12}$. How many intermediate fields F are there with
   (a) $[F : K] = 4$?
   (b) $[F : K] = 9$?
   (c) $\mathrm{Gal}(L/F) \simeq \mathbb{Z}_4$?

10. Factor $x^4 - 2$ into irreducible polynomials over each of the fields
    (a) $\mathbb{Q}$
    (b) $\mathbb{Q}(\sqrt{2})$
    (c) $\mathbb{Q}(i)$
    (d) $\mathbb{Q}(\sqrt{2}, i)$
    (e) $\mathbb{Q}(\sqrt[4]{2})$
    (f) $\mathbb{Q}(\sqrt[4]{2}, i)$
    (g) $\mathbb{F}_4$

11. Factor $x^{16} - x$ in $\mathbb{F}_4$ and $\mathbb{F}_8$.

12. Factor $x^{27} - x$ in $\mathbb{F}_3$. Determine its Galois group over $\mathbb{F}_3$ and all the intermediate field extensions.

13. (a) Determine the splitting field K of the polynomial $X^5 - 2$ over $\mathbb{Q}$.
    (b) Determine the degree $[K : \mathbb{Q}]$ and the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$.
    (c) Determine all the subgroups of G.
    (d) Determine the lattice of the subgroups of G.
    (e) For each subgroup H of G determine the fixed field K^H of H.
    (f) Determine the lattice of the subfields of K.
    (g) Determine the subfields $F \subseteq K$ that are normal extensions of $\mathbb{Q}$.

14. (a) Determine the splitting field K of the polynomial $X^9 - 1$ over $\mathbb{Q}$.
    (b) Determine the degree $[K : \mathbb{Q}]$ and the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$.
    (c) Determine all the subgroups of G.

(d) Determine the lattice of the subgroups of G.
(e) For each subgroup H of G determine the fixed field K^H of H.
(f) Determine the lattice of the subfields of K.
(g) Determine the subfields $F \subseteq K$ that are normal extensions of $\mathbb{Q}$.

15. (a) Determine the splitting field K of the polynomial $X^9 - 1$ over $F = \mathbb{F}_3$.
(b) Determine the degree $[K : F]$ and the Galois group $G = \mathrm{Gal}(K/F)$.
(c) Determine all the subgroups of G.
(d) Determine the lattice of the subgroups of G.
(e) For each subgroup H of G determine the fixed field K^H of H.
(f) Determine the lattice of the subfields of K.
(g) Determine the subfields $L \subseteq K$ that are normal extensions of F.

16. (a) Prove that the polynomial $f(x) = x^4 - 8x^2 + 11$ is irreducible.
(b) Determine all the intermediate fields when K is the splitting field of $f(x)$ over $\mathbb{Q}$.
(c) Determine the subfields $L \subseteq K$ that are normal extensions of $\mathbb{Q}$.

17. Determine the Galois group of the following polynomials over $\mathbb{Q}$:
(a) $x^3 - 21x + 7$
(b) $x^3 + x^2 - 2x - 1$
(c) $x^3 + x^2 - 2x + 1$
(d) $x^4 + 4x^2 + 2$
(e) $x^4 + 2x^2 + 4$
(f) $x^4 + 1$
(g) $x^9 - 1$
(h) $x^{12} - 1$

18. Determine the degree of ζ_7 over $\mathbb{Q}(\zeta_3)$. (Recall that the degree of α over K is $[K(\alpha) : K]$.

19. Determine the degree of the following elements over $\mathbb{Q}$:
(a) $\zeta_7 + \zeta_7^5$
(b) $\zeta_7^3 + \zeta_7^5 + \zeta_7^6$

20. Determine explicitly all intermediate field extensions in $\mathbb{Q}(\zeta_{11})/\mathbb{Q}$.

21. Determine explicitly all intermediate field extensions in $\mathbb{Q}(\zeta_{12})/\mathbb{Q}$.

22. Determine the centralizer and the order of the conjugacy class of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{F}_3)$ and $\mathrm{SL}_2(\mathbb{F}_3)$.

23. Determine the centralizer and the order of the conjugacy class of (12) in S_5.

24. Determine the orders of the elements in S_7.

25. How many elements of order 5 might be contained in a group of order 20?

26. Find 2-Sylow subgroups of the following groups:
(a) D_4

(b) $\mathbb{Z}_{12}$

(c) S_4

27. Show that there are no simple groups of order pq where p and q are primes (not necessarily distinct) or give a counterexample and salvage if possible.

28. Show that there are no simple groups of order 144.

29. Show that any group of order p^2q is solvable, where p and q are primes (not necessarily distinct).

30. Consider the Frobenius group

$$F_{20} = \left\{ \begin{pmatrix} m & n \\ 0 & 1 \end{pmatrix} ; m \in \mathbb{Z}_5^\times, n \in \mathbb{Z}_5 \right\}.$$

(a) Find a composition series for F_{20}.

(b) Find the descending series of commutator subgroups of F_{20}.

31. Show that the action via permutations of A_4 on $\{1, 2, 3, 4\}$ is transitive.

32. (a) Show that $G = \mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H} = \{z \in \mathbb{C}; \mathrm{Im}(z) > 0\}$ via

$$\gamma \cdot z = \frac{az + b}{cz + d} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

(b) Find the stabilizer of i with respect to this group action.

(c) Show that the action is transitive.