HW # 5 (1)

$\boxed{7.3 \# 2}$

(a) • $\ker(\Phi)$ is normal for free.

• $\ker(\Phi) \subseteq H$: if $x \in \ker(\Phi)$, then
$x \cdot H = H \Rightarrow x \in H$.

• Suppose $N \trianglelefteq G$ and $N \subseteq H$. We'll show that $N \subseteq \ker(\Phi)$.

Indeed, let $x \in N$. N normal $\Rightarrow g^{-1}xg \in N$ $\forall g \in G \Rightarrow g^{-1}xg \in H$ $\forall g \in G \Rightarrow x(gH) = gH$ $\forall g \in G$

But this is what it means for x to be in the kernel of Φ.

(b) $[G:H] = n \Rightarrow |S| = n$. Hence $|\mathrm{Sym}(S)| = n!$

By the 1^{st} homomorphism theorem,

$$\frac{|G|}{|\ker(\Phi)|} \,\Big|\, |\mathrm{Sym}(S)|.$$

So $|G| \nmid \underbrace{|\mathrm{Sym}(S)|}_{n!} \Rightarrow |\ker(\Phi)| > 1$.

Thus, $\ker(\Phi)$ is a <u>nontrivial</u> normal subgroup of G. (By (a), $\ker(\Phi) \subseteq H$.)

$\boxed{7.3\#5}$ By Cauchy's Thm (7.2.10), G has a subgroup $H \leq G$ of order 7.
$\Rightarrow [G:H] = 4$.
But $\underbrace{28}_{|G|} \nmid \underbrace{24}_{4!}$, so exercise #2
implies H contains a nontrivial normal subgroup of G. Since $|H|=7$, its only nontrivial subgroup is all of H.
$\Rightarrow H \trianglelefteq G$, $|H| = 7$.

If $K \trianglelefteq G$, $|K| = 4$ by Thm 7.1.3,
$G \cong H \times K$.

- $|H| = 7 \Rightarrow$ cyclic $\Rightarrow$ abelian
- $|K| = 4 \Rightarrow K = \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \Rightarrow$ abelian.

So G is abelian.

③

$\boxed{7.3^{\#7}}$ We use induction on $|G|$.

- $|G|=p$ — nothing to prove.
- Assume the result holds for all $|G'|=p^k$, $1\le k\le n$.

We want to show it also holds for $|G|=p^{n+1}$.

By Burnside (Thm 7.2.8), $e \neq Z(G)$.
So let $H \le Z(G)$ have order p.
Note that $H \trianglelefteq G$.

then consider
$$\rho: G \longrightarrow G/H.$$

Since $|G/H|=p^n$, for any $0<k<n$, there's a normal subgroup of order p^k in G/H by our induction hypothesis.

~~Theorem~~ Prop 3.8.7 then implies that these subgroups of G/H correspond to normal subgroups of G of orders p^{k+1}, $0<k<n$.

As $H \trianglelefteq G$ has order p (the only one we're missing), we have the result. $\square$

$\boxed{7.3 \# 11}$ Lemma 7.3.7 says that $|S| \equiv |S^G| \pmod p$. So $p \nmid |S|$ $\Rightarrow p \nmid |S^G| \Rightarrow |S^G| \neq 0$.

$\boxed{7.3 \# 13}$

a) For any pair $a, b \in \{1, \dots, n\}$, we see that $(a\ b)\, a = b$.

b) Let $a, b, c \in \{1, \dots, n\}$ be distinct. Then $(b\ c)(a\ c)\, a = b$.

c) Let $v, w \in V$ be vectors.

<u>Case 1</u>: If $w = \alpha v$ for some $\alpha \in F$, then we note $(\alpha I) v = w$ and $(\alpha I) \in GL_n(F)$.

<u>Case 2</u>: If w, v are linearly independent, ~~let~~ pick a basis of V containing v, w ~~w, v~~: $\{ v, w, e_3, \dots, e_n \}$.

~~Then let~~ In this basis, $v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ and $w = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

~~Note~~

$$\underbrace{\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline & & \begin{matrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{matrix} \\ & 0 & \end{array}\right)}_{g} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{v} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{w}$$

and $g \in GL_n(F)$.

(5)

d) $S = G\cdot x$ for any $x \in S$.

$\Rightarrow |S| = |G\cdot x| = [G : G_x]$ by orbit-stabilizer.

If $|S| > 1$, then for any $s \in S$, there exists some $t \in S$ with $s \neq t$ and a $g \in G$ satisfying $g(s) = t$. $\Rightarrow s \notin S^G$.

$\boxed{7.4\#1}$ G abelian $\Rightarrow$ G has only 1 sylow-p subgroup. Call it K.

Let $S = \{e\} \cup \{g \in G \mid ord(g) = p^k \text{ for some } k\}$.

We want to show $S = K$.

- Clearly $S \supseteq K$.
- Conversely, let $x \in S$. Then $\langle x \rangle$ is a p-group

$\Rightarrow \langle x \rangle$ is contained in a Sylow p-subgroup.

$\Rightarrow \langle x \rangle \leq K$

$\Rightarrow x \in K$. $\square$

$\boxed{7.4 \# 3}$ $|S_4| = 24$.

- Sylow-3 subgroup has order 3, so $\langle (1\,2\,3) \rangle$ works.
- Sylow-2 subgroup has order 8, we can show $\langle (1\,2\,3\,4), (1\,3) \rangle$ works.

$\boxed{7.4 \# 4}$

D_5 — see problem #5.

$|D_6| = 12 = 2^2 \cdot 3$.

Let $D_6 = \langle \sigma, \tau \mid \sigma^6 = \tau^2 = e,\ \sigma\tau = \tau\sigma^{-1} \rangle$.

Calculate that:

- Sylow-2's are $\langle \sigma^3, \tau \rangle$, $\langle \sigma^3, \sigma\tau \rangle$, $\langle \sigma^3, \sigma^2\tau \rangle$.
- Sylow 3 is $\langle \sigma^2 \rangle$.

$\boxed{7.4 \# 5}$

$|D_{p^k}| = 2 \cdot p^k$.

$D_{p^k} = \langle \sigma, \tau \mid \sigma^{p^k} = \tau^2 = e,\ \sigma\tau = \tau\sigma^{-1} \rangle$.

Calculate that: —

- Sylow-2's are $\langle \sigma^i \tau \rangle$ for $0 \leq i < p^k$.
- ~~Sylow 3's are~~ Sylow p is $\langle \sigma \rangle$.

$\boxed{7.4 \# 6}$ Solution is in back of book.