HW #4.

$\boxed{1.}$ Let $p: G \to G/H$ by $p(g) = gH$.

Suppose $f: G \to G'$ with $H < \ker(f)$. We want to show there exists a unique (group) homomorphism $\bar{f}: G/H \to G'$ such that $f = \bar{f} \circ p$; ie so that

$$\begin{tikzcd} G \arrow[r, "p"] \arrow[dr, "f"'] & G/H \arrow[d, dashed, "\bar{f}"] \\ & G' \end{tikzcd}$$

commutes.

Define $\bar{f}: G/H \to G'$ by $\bar{f}(gH) = f(g)$.

<u>well defined</u>: Suppose $gH = g'H$. We want $f(g) = f(g')$.

$gH = g'H \Rightarrow g^{-1}g' \in H$, so write $g^{-1}g' = h \in H$.

So $g' = gh$. Apply f:-

$$f(g') = f(gh) = f(g)f(h)$$
$$= f(g) \quad // \quad \text{since } h \in H \leq \ker(f)$$

<u>Homomorphism</u>: Want $\bar{f}(gHg'H) = \bar{f}(gH)\,\bar{f}(g'H)$.

$\bar{f}(gHg'H) = \bar{f}(gg'H) = f(gg')$

$\bar{f}(gH)\,\bar{f}(g'H) = \cancel{gHg'H} =$

$= f(g)f(g') = f(gg')$ since f is a homomorphism. //

<u>Unique</u> Suppose $\tilde{f} : G/H \to G'$ also satisfies that $\tilde{f} \circ p = f$.

Then $f(g) = \tilde{f} \circ p(g) = \tilde{f}(gH)$.

But this is exactly how $\bar{f}$ is defined! $\square$

$\boxed{2}$ Let $p : R \to R/I$ where $I \subseteq R$ is an ideal. Suppose $f : R \to S$ with $I \subseteq \ker(f)$.

We want to prove :—

there exists a unique ring homomorphism $R/I \xrightarrow{\bar{f}} S$ such that $\bar{f} \circ p = f$.

$$\begin{tikzcd} R \arrow[r, "p"] \arrow[dr, "f"'] & R/I \arrow[d, "\bar{f}"] \\ & S \end{tikzcd}$$

③

Define $\bar{f}: R/I \to S$ by $\bar{f}(r+I) = f(r)$.

<u>Well defined</u> If $r+I = r'+I$, then $\exists\, x \in I$ with $r = r'+x$. Apply f to get

$$f(r) = f(r'+x) = f(r') + f(x)$$
$$= f(r') \quad \text{as } x \in I \subseteq \ker(f).$$

<u>Ring homomorphism</u>

- $\bar{f}(1+I) = f(1) = 1.$
- $\bar{f}((r+I)+(r'+I)) = \bar{f}((r+r')+I) = f(r+r') = f(r)+f(r')$
  $= \bar{f}(r+I) + \bar{f}(r'+I).$
- $\bar{f}((r+I)(r'+I)) = \bar{f}(rr'+I) = f(rr') = f(r)f(r')$
  $= \bar{f}(r+I)\,\bar{f}(r'+I).$

<u>Unique</u> If $\tilde{f}: R/I \to S$ satisfies $\tilde{f} \circ p = f$, then $f(r) = \tilde{f} \circ p(r) = \tilde{f}(r+I)$, which is exactly how $\bar{f}$ is defined. $\square$

3 Let R be a commutative ring. Let $f: R \to S$, where S is a commutative ring, and let $s_0 \in S$. Then there exists a unique homom. ~~map~~ $\bar{f}: R[x] \to S$ satisfying $f = \bar{f} \circ i$ and $\bar{f}(x) = s_0$, where $i: R \to R[x]$ is inclusion.

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[dr, "f"'] & R[x] \arrow[d, "\bar{f}"] \\ & S \end{tikzcd}$$

Indeed, $\bar{f}(a_n x^n + \cdots + a_0) := f(a_n) s_0^n + \cdots + f(a_0)$.

Easy to check $\bar{f}$ is a ring homomorphism. (but I'm being lazy.)

<u>Uniqueness</u>: If $\tilde{f}: R[x] \to S$ satisfies $f = \tilde{f} \circ i$ and $\tilde{f}(x) = s_0$, then we notice

$$f(r) = \tilde{f} \circ i(r) = \tilde{f}(r) \text{ for all } r \in R.$$

Hence
$$\tilde{f}(a_n x^n + \cdots + a_0) = \tilde{f}(a_n)\tilde{f}(x)^n + \cdots + \tilde{f}(a_0)$$
$$= f(a_n) s_0^n + \cdots + f(a_0) \quad \square$$

$\boxed{4}$. (a) First convince yourself $(\cdot)^\sigma$ is a ring homomorphism.

It's an automorphism because it has an inverse, namely $(\cdot)^{\sigma^{-1}}$ where σ^{-1} is the automorphism of L inverse to σ.

Indeed,

$$\left((a_nX^n + \cdots + a_0)^\sigma\right)^{\sigma^{-1}} = \left(\sigma(a_n)X^n + \cdots + \sigma(a_0)\right)^{\sigma^{-1}}$$
$$= \sigma^{-1}(\sigma(a_n))X^n + \cdots + \sigma^{-1}(\sigma(a_0))$$
$$= a_nX^n + \cdots + a_0.$$

We can view $L \subseteq L[x]$ (constant polynomials).

for $a \in L$, we notice

$$\underset{\substack{\uparrow \\ \text{viewed as a} \\ \text{constant poly.}}}{(a)}^\sigma = \sigma(a),$$

Thus, $(\cdot)^\sigma$ and σ do the same thing to L;

ie. $(\cdot)^\sigma$ extends σ.

(b) Let $f \in L[X]$. Show $f \in K[X] \iff f^\sigma = f \ \forall \sigma \in Gal(L/K)$.

Pf: $\Rightarrow$] Every coefficient of f is fixed by every element of $Gal(L/K)$.

$\Leftarrow$] Each coeff. a_i of f is fixed by every element of $Gal(L/K)$ $\Rightarrow$ $a_i \in L^{Gal(L/K)} \ \forall i$.

But L/K Galois $\Rightarrow$ $L^{Gal(L/K)} = K$. $\square$

$\boxed{5}$ (a) G acts on X if there is a group homomorphism $G \longrightarrow Sym(X)$.

(b) orbit: $\mathcal{O}_G(x) = \{ y \in X \mid g \cdot x = y \text{ for some } g \in G \}$.

stabilizer: $Stab_G(x) = \{ g \in G \mid g \cdot x = x \}$.

6 (a) The key is that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$,

as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & -ab+ab \\ cd-dc & -bc+ad \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. //

(b) $\left(\dfrac{az+b}{cz+d}\right)\left(\dfrac{c\bar{z}+d}{c\bar{z}+d}\right) = \dfrac{ac|z|^2 + ad\,z + bc\bar{z} + bd}{|cz+d|^2}$

The imaginary part is $\dfrac{ad\,\mathrm{Im}(z) - bc\,\mathrm{Im}(z)}{|cz+d|^2} = \dfrac{\mathrm{Im}(z)}{|cz+d|^2}$. //

(c) This follows from part (b).

(d) Need to show that

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right]\cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\left[\begin{pmatrix} e & f \\ g & h \end{pmatrix}\cdot z\right]$$

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right]\cdot z = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}\cdot z = \frac{(ae+bg)z+(af+bh)}{(ce+dg)z+(cf+dh)}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left[\begin{pmatrix} e & f \\ g & h \end{pmatrix} \cdot z \right] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{ez+f}{gz+h}$$

$$= \frac{a\left(\frac{ez+f}{gz+h}\right) + b}{c\left(\frac{ez+f}{gz+h}\right) + d}$$

$$= \frac{aez+af+bgz+bh}{cez+cf+dgz+dh}.$$

(e) $\mathrm{Stab}_G(i) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \;\middle|\; \frac{ai+b}{ci+d} = i \right\}.$

But $\displaystyle \frac{ai+b}{ci+d} = \left(\frac{ai+b}{ci+d}\right)\left(\frac{-ci+d}{-ci+d}\right)$

$$= \frac{ac+bd+(ad-bc)i}{c^2+d^2}$$

$$= \frac{ac+bd+i}{c^2+d^2}.$$

So $c^2+d^2=1$. and $ac+bd=0$.

$c=\pm 1,\ d=0 \Rightarrow a=0. \Rightarrow b=\mp 1$

$c=0,\ d=\pm 1 \Rightarrow b=0 \Rightarrow a=\pm 1$

So $\mathrm{Stab}_G(i) = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$