

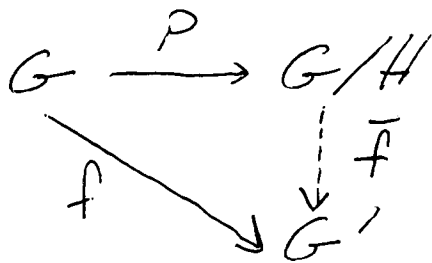
# HW #4.

(1)

1. Let  $p: G \rightarrow G/H$  by  $p(g) = gH$ .

Suppose  $f: G \rightarrow G'$  with  $H \subseteq \ker(f)$ . We want to show there exists a unique <sup>(group)</sup> homomorphism

$\bar{f}: G/H \rightarrow G'$  such that  $f = \bar{f} \circ p$ ; i.e. so that



commutes.

Define  $\bar{f}: G/H \rightarrow G'$  by  $\bar{f}(gH) = f(g)$ .

well defined: Suppose  $gH = g'H$ . we want  $f(g) = f(g')$ .

$gH = g'H \Rightarrow g^{-1}g' \in H$ , so write  $g^{-1}g' = h \in H$ .

so  $g' = gh$ . Apply  $f$ : -

$$\begin{aligned} f(g') &= f(gh) = f(g)f(h) \\ &= f(g) \quad \text{since } h \in H \subseteq \ker(f). \end{aligned}$$

(2)

Homomorphism: Want  $\bar{f}(gHg^{-1}) = \bar{f}(gH)\bar{f}(g^{-1}H)$ .

$$\bar{f}(gHg^{-1}) = \bar{f}(gg^{-1}H) = f(gg^{-1})$$

$$\bar{f}(gH)\bar{f}(g^{-1}H) = \cancel{gHg^{-1}H} = f(g)f(g^{-1}) = f(gg^{-1}) \text{ since } f \text{ is a homomorphism.} //$$

Unique Suppose  $\tilde{f}: G/H \rightarrow G'$  also satisfies that  $\tilde{f} \circ \rho = f$ .

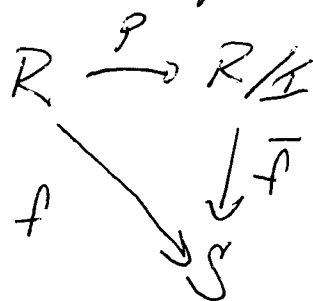
$$\text{Then } f(g) = \tilde{f} \circ \rho(g) = \tilde{f}(gH).$$

But this is exactly how  $\bar{f}$  is defined!  $\square$

[2] Let  $\rho: R \rightarrow R/I$  where  $I \subseteq R$  is an ideal. Suppose  $f: R \rightarrow S$  with  $I \subseteq \ker(f)$ .

We want to prove: —

there exists a unique ring homomorphism  $\bar{f}: R/I \rightarrow S$  such that  $\bar{f} \circ \rho = f$ .



③

Define  $\bar{f}: R/I \rightarrow S$  by  $\bar{f}(r+I) = f(r)$ .

Well defined If  $r+I = r'+I$ , then  $\exists x \in I$  with

$r = r' + x$ . Apply  $f$  to get

$$f(r) = f(r' + x) = f(r') + f(x)$$

$$= f(r') \quad \text{as } x \in I \subseteq \ker(f). //$$

Ring homomorphism

$$\circ \bar{f}(1+I) = f(1) = 1.$$

$$\circ \bar{f}((r+I) + (r'+I)) = \bar{f}((r+r') + I) = f(r+r') = f(r) + f(r') \\ = \bar{f}(r+I) + \bar{f}(r'+I).$$

$$\circ \bar{f}((r+I)(r'+I)) = \bar{f}(rr' + I) = f(rr') = f(r)f(r') \\ = \bar{f}(r+I)\bar{f}(r'+I). //$$

Unique If  $\tilde{f}: R/I \rightarrow S$  satisfies  $\tilde{f} \circ \rho = f$ , then

$$f(r) = \tilde{f} \circ \rho(r) = \tilde{f}(r+I), \quad \text{which is exactly}$$

how  $\bar{f}$  is defined.



(4)

[3] Let  $R$  be a commutative ring. Let

$f: R \rightarrow S$ , where  $S$  is a commutative ring,  
and let  $s_0 \in S$ . Then there exists a unique  
homom.  $\bar{f}: R[x] \rightarrow S$  satisfying  $f = \bar{f} \circ i$  and  
 $\bar{f}(x) = s_0$ , where  $i: R \rightarrow R[x]$  is inclusion.

$$\begin{array}{ccc} R & \xrightarrow{i} & R[x] \\ & \searrow f & \downarrow \bar{f} \\ & & S \end{array}$$

Indeed,  $\bar{f}(a_n x^n + \dots + a_0) := f(a_n) s_0^n + \dots + f(a_0)$ .

Easy to check  $\bar{f}$  is a ring homomorphism. (but I'm being lazy.)

Uniqueness: If  $\tilde{f}: R[x] \rightarrow S$  satisfies  $f = \tilde{f} \circ i$  and

$\tilde{f}(x) = s_0$ , then we notice

$$f(r) = \tilde{f} \circ i(r) = \tilde{f}(r) \quad \text{for all } r \in R.$$

$$\begin{aligned} \text{Hence } \tilde{f}(a_n x^n + \dots + a_0) &= \tilde{f}(a_n) \tilde{f}(x)^n + \dots + \tilde{f}(a_0) \\ &= f(a_n) s_0^n + \dots + f(a_0) \quad \square \end{aligned}$$

(5)

4. (a) First convince yourself  $(\cdot)^\sigma$  is a ring homomorphism. It's an automorphism because it has an inverse, *exactly*  $(\cdot)^{\sigma^{-1}}$  where  $\sigma^{-1}$  is the automorphism of  $L$  inverse to  $\sigma$ .

Indeed,

$$\begin{aligned} ((a_n x^n + \dots + a_0)^\sigma)^\sigma &= (\sigma(a_n) x^n + \dots + \sigma(a_0))^\sigma \\ &= \sigma^{-1}(\sigma(\sigma(a_n))) x^n + \dots + \sigma^{-1}(\sigma(\sigma(a_0))) \\ &= a_n x^n + \dots + a_0. \end{aligned}$$

We can view  $L \subseteq L[x]$  (constant polynomials).

For  $a \in L$ , we notice

$$(a)^\sigma = \sigma(a),$$

↑  
viewed as a  
constant poly.

Thus,  $(\cdot)^\sigma$  and  $\sigma$  do the same thing to  $L$ ;

i.e.  $(\cdot)^\sigma$  extends  $\sigma$ .

(6)

(b) Let  $f \in L[X]$ . Show  $f \in K[X] \Leftrightarrow f^\sigma = f \forall \sigma \in \text{Gal}(L/K)$ .

Pf:  $\Rightarrow$  Every coefficient of  $f$  is fixed by every element of  $\text{Gal}(L/K)$ .

$\Leftarrow$  Each coeff.  $a_i$  of  $f$  is fixed by every element of  $\text{Gal}(L/K) \Rightarrow a_i \in L^{\text{Gal}(L/K)} \forall i$ .

But  $L/K$  Galois  $\Rightarrow L^{\text{Gal}(L/K)} = K$ .  $\square$

[5] (a)  $G$  acts on  $X$  if there is a group homomorphism  $G \rightarrow \text{Sym}(X)$ .

(b) orbit:  $O_G(x) = \{y \in X \mid g \cdot x = y \text{ for some } g \in G\}$ .

stabilizer:  $\text{Stab}_G(x) = \{g \in G \mid g \cdot x = x\}$ .

(17)

[6] (a) The key is that  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ ,

$$\text{as } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & -ab + ab \\ cd - dc & -bc + ac \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} //$$

$$(b) \left( \frac{az+b}{cz+d} \right) \left( \frac{c\bar{z}+d}{c\bar{z}+d} \right) = \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz+d|^2}$$

$$\text{The imaginary part is } \frac{ad \operatorname{Im}(z) - bc \bar{z} \operatorname{Im}(z)}{|cz+d|^2} = \frac{\operatorname{Im}(z)}{|cz+d|^2} //$$

(c) This follows from part (b).

(d) Need to show that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left[ \begin{pmatrix} e & f \\ g & h \end{pmatrix} \cdot z \right]$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \cdot z = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} \cdot z = \frac{(ae + bg)z + (af + bh)}{(ce + dg)z + (cf + dh)}$$

$$\begin{aligned}
 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} \cdot z &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{ez+f}{gz+h} \\
 &= \frac{a\left(\frac{ez+f}{gz+h}\right) + b}{c\left(\frac{ez+f}{gz+h}\right) + d} \\
 &= \frac{aez+af + bgz+bh}{cez+cf + dgz+dh}. \quad \checkmark
 \end{aligned}$$

$$(e) \text{ Stab}_G(i) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \mid \frac{az+b}{cz+d} = i \right\}$$

$$\begin{aligned}
 \text{But } \frac{az+b}{cz+d} &= \left( \frac{az+b}{cz+d} \right) \left( \frac{-ci+d}{-ci+d} \right) \\
 &= \frac{ac + bd + (ad - bc)i}{c^2 + d^2} \\
 &= \frac{ac + bd + i}{c^2 + d^2}
 \end{aligned}$$

$$\text{So } c^2 + d^2 = 1, \text{ and } ac + bd = 0.$$

$$c = \pm 1, d = 0 \Rightarrow a = 0, \Rightarrow b = \mp 1$$

$$c = 0, d = \pm 1 \Rightarrow b = 0 \Rightarrow a = \pm 1$$

$$\text{So } \text{Stab}_G(i) = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$