HOMEWORK 4

DUE 1 MAY 2013

1. Let G be a group and H be a normal subgroup of G. Prove the universality property of the quotient group G/H :

If $f : G \to G'$ is a group homomorphism such that $H \subset \ker f$, there exists a unique group homomorphism $\bar{f} : G/H \to G'$ such that $\bar{f} \circ p = f$ where $p : G \to G/H$ is the canonical projection.

2. Formulate and prove the universality property for the quotient ring. *Attention! The initial ring has no reason to be commutative.*

3. Formulate and prove the universality property for the ring of polynomials $R[X]$ over a commutative ring R.

4. Let L/K be a finite Galois extension. For each automorphism σ of L, consider the natural extension $(\cdot)^\sigma : L[X] \to L[X]$ given by $(a_nX^n + \cdots + a_0)^\sigma = \sigma(a_n)X^n + \cdots + \sigma(a_0)$.

(a) Show that $(\cdot)^\sigma : L[X] \to L[X]$ is a ring automorphism that extends the original field automorphism $\sigma : L \to L$.

(b) Let $f \in L[X]$. Show that $f \in K[X] \iff f^\sigma = f$ for all $\sigma \in \mathrm{Gal}(L/K)$.

5. (a) Give the definition of an action of a group G on a set X.
(b) Define the orbit and stabilizer of an element $x \in X$ with respect to the action of G.

6. Let

$$G = \mathrm{SL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

and $\mathcal{H} = \{z \in \mathbb{C}; \mathrm{Im}(z) > 0\}$ the complex upper half-plane.

(a) Show that G is a group.

(b) If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $z \in \mathbb{C}$ show that

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

(c) Show that $\dfrac{az+b}{cz+d} \in \mathcal{H}$ for all $z \in \mathcal{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

(d) Show that

$$G \times \mathcal{H} \to \mathcal{H}, (\gamma, z) \to \gamma \cdot z = \frac{az+b}{cz+d}, \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a group action.

(e) Find the stabilizer of i with respect to this group action.