

HW #3

①

1(a). From HW#2,

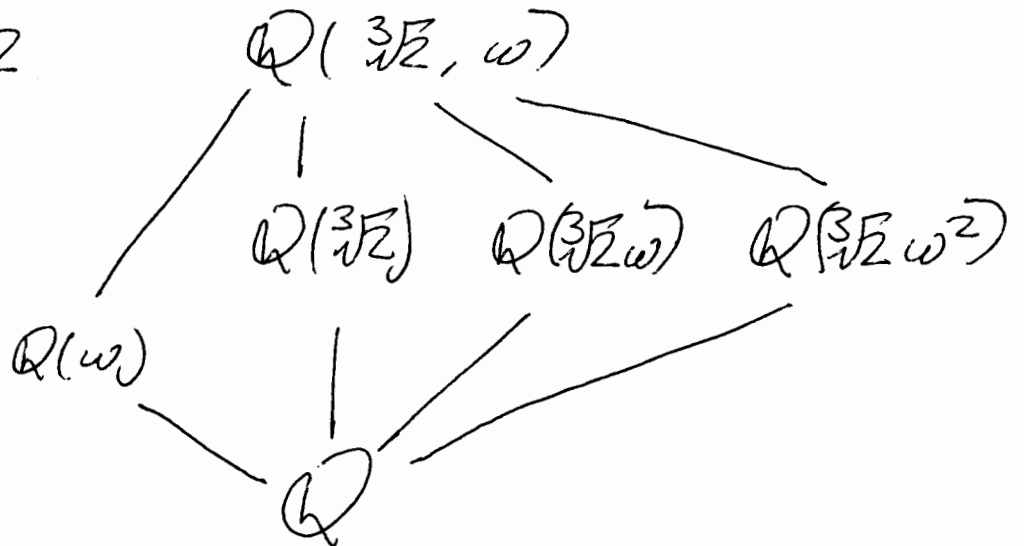
$$\begin{array}{c} \mathbb{F}_{3^{16}} = L \\ | \\ \mathbb{F}_{3^8} \\ | \\ \mathbb{F}_{3^4} \\ | \\ \mathbb{F}_{3^2} = K \end{array}$$

(b). $\mathbb{F}_{3^{16}}$ is the splitting field of $x^{3^{16}} - x = 0$ over ~~any finite~~ $\mathbb{F}_{3^2}, \mathbb{F}_{3^4}, \mathbb{F}_{3^8}, \mathbb{F}_{3^{16}}$. \Rightarrow all of them.

(c). F/K is normal $\Leftrightarrow \text{Gal}(L/F) \trianglelefteq \text{Gal}(L/K)$.

Since $\text{Gal}(L/K) \cong \mathbb{Z}_8$ is abelian, every subgroup is normal. \Rightarrow all of them.

2(a) From HW#2



(2)

(b) $\mathbb{Q}(\sqrt[3]{2}, \omega)$ is the splitting field of $x^3 - 2$ over all of its subfields \Rightarrow all of them.

(c) K/\mathbb{Q} is normal $\Leftrightarrow \text{Gal}(\mathbb{Q}(\sqrt[3]{2}, \omega)/K)$ is a normal subgroup of $\text{Gal}(\mathbb{Q}(\sqrt[3]{2}, \omega), \mathbb{Q})$.

The subgroups which aren't normal are

$$\langle \phi \rangle, \langle \theta \phi \rangle, \langle \theta^2 \phi \rangle$$

since $\theta(\phi)\theta^{-1} = \phi\theta^{-2}(\theta^2) = \phi\theta = \theta^2\phi$.

and $\theta(\theta\phi)\theta^{-1} = \theta\phi\theta^2\theta^2 = \theta\phi\theta = \phi$.

Hence, ~~at~~ every subfield of $\mathbb{Q}(\sqrt[3]{2}, \omega)$ is normal

over \mathbb{Q} except $\mathbb{Q}(\sqrt[3]{2}), \mathbb{Q}(\sqrt[3]{2}\omega), \mathbb{Q}(\sqrt[3]{2}\omega^2)$.

(3)

8.3#1

Note $\alpha^3 \beta (\sqrt[4]{2} - i \sqrt[4]{2}) = \alpha^3 (\sqrt[4]{2} + i \sqrt[4]{2})$

$$= \alpha (\sqrt[4]{2} i + i \sqrt[4]{2} i)$$

$$= \alpha (-\sqrt[4]{2} - i \sqrt[4]{2})$$

$$= (-\sqrt[4]{2} i + \sqrt[4]{2}) \checkmark$$

So $\mathbb{Q}(\sqrt[4]{2} - i \sqrt[4]{2}) \subseteq \mathbb{Q}(\sqrt[4]{2}, i)^{\langle \alpha^3 \beta \rangle}$

But since $[\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}(\sqrt[4]{2} - i \sqrt[4]{2})] = |\langle \alpha^3 \beta \rangle|$,
by the Fundamental Theorem,

$$\mathbb{Q}(\sqrt[4]{2} - i \sqrt[4]{2}) = \mathbb{Q}(\sqrt[4]{2}, i)^{\langle \alpha^3 \beta \rangle}$$

8.3#2

Recall that conjugate subfields correspond to conjugate subgroups. Moreover, all subgroups of index 2 are normal, so we only have to consider the subgroups

$$\langle \beta \rangle, \langle \alpha^2 \beta \rangle, \langle \alpha^2 \rangle, \langle \alpha \beta \rangle, \langle \alpha^3 \beta \rangle$$

We also note that the center of G is

$$\langle \alpha^2 \rangle \Rightarrow \langle \alpha^2 \rangle \text{ is normal.}$$

Moreover, $\langle \alpha^2 \rangle \leq \text{Stab}_G(g)$ for any $g \in G$.

So $\langle \beta, \alpha^2 \rangle \leq \text{Stab}_G(\beta) \Rightarrow \beta$ has at most $\frac{|G|}{|\langle \beta, \alpha^2 \rangle|} = 2$ elt in its orbit (by Orbit-Stabilizer Thm).

Since $\alpha\beta\alpha^{-1} = \alpha^2\beta$, we see that

$\langle \beta \rangle$ is conjugate to $\langle \alpha^2\beta \rangle$ by α .

$\Rightarrow \mathbb{Q}(\sqrt[4]{2})$ is conj. to $\mathbb{Q}(i\sqrt[4]{2})$ by α .

Similarly, we can conclude that $\langle \alpha\beta \rangle$ is only conjugate to $\langle \alpha^3\beta \rangle$ (and itself), where

$$\beta(\alpha\beta)\beta^{-1} = \alpha^3\beta.$$

$\Rightarrow \mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2})$ is conj. to $\mathbb{Q}(\sqrt[4]{2} - i\sqrt[4]{2})$ by β .

8.3 #3 The roots of x^4+1 are $\frac{1}{\sqrt{2}}(\pm 1 \pm i)$, so

the splitting field is $\mathbb{Q}(\sqrt{2}, i)$. The Galois

group is $\mathbb{Z}_2 \times \mathbb{Z}_2$, generated by

$$\alpha: \begin{array}{l} \sqrt{2} \mapsto -\sqrt{2} \\ i \mapsto i \end{array}$$

$$\beta: \begin{array}{l} \sqrt{2} \mapsto \sqrt{2} \\ i \mapsto -i. \end{array}$$

8.3#4

The roots of $x^4 - x^2 - 6 = (x^2 - 3)(x^2 + 2)$

are $\pm\sqrt{3}$ and $\pm i\sqrt{2}$. So the S.F. is $\mathbb{Q}(\sqrt{3}, i\sqrt{2})$.

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$$\alpha: \sqrt{3} \mapsto -\sqrt{3} \\ i\sqrt{2} \mapsto i\sqrt{2}$$

$$\beta: \sqrt{3} \mapsto \sqrt{3} \\ i\sqrt{2} \mapsto -i\sqrt{2}$$

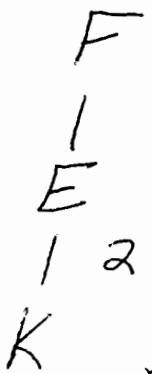
8.3#5

The roots of $x^8 - 1$ over \mathbb{Q} are

$\pm 1, \pm i, \frac{1}{\sqrt{2}}(\pm 1 \pm i)$. So the splitting field

is $\mathbb{Q}(\sqrt{2}, i)$ as in Ex 8.3#3.

8.3#6



$$[E:K] = 2 \Rightarrow [\text{Gal}(F/K) : \text{Gal}(F/E)] = 2$$

$$\Rightarrow \text{Gal}(F/E) \trianglelefteq \text{Gal}(F/K)$$

Thus E is normal over K .

8.3#7

See, for example, Dr. PMH Wilson's

notes on Galois Theory at

www.jchl.co.uk/maths/galois.pdf