HW #3

1(a). From HW#2,

$$\begin{tikzcd}
\mathbb{F}_{3^{16}} = L \arrow[d, dash] \\
\mathbb{F}_{3^8} \arrow[d, dash] \\
\mathbb{F}_{3^4} \arrow[d, dash] \\
\mathbb{F}_{3^2} = K
\end{tikzcd}$$

(b). $\mathbb{F}_{3^{16}}$ is the splitting field of $x^{3^{16}} - x = 0$ over ~~any field~~ $\mathbb{F}_{3^2}$, $\mathbb{F}_{3^4}$, $\mathbb{F}_{3^8}$, $\mathbb{F}_{3^{16}}$. $\Rightarrow$ $\boxed{\text{all of them.}}$

(c). F/K is normal $\Leftrightarrow$ $Gal(L/F) \trianglelefteq Gal(L/K)$.

Since $Gal(L/K) \cong \mathbb{Z}_8$ is abelian, every subgroup is normal. $\Rightarrow$ $\boxed{\text{all of them}}$

2 (a) From HW #2

$$\begin{tikzcd}
& & \mathbb{Q}(\sqrt[3]{2},\omega) \arrow[dll, dash] \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & & \\
\mathbb{Q}(\omega) \arrow[drr, dash] & & \mathbb{Q}(\sqrt[3]{2}) \arrow[d, dash] & \mathbb{Q}(\sqrt[3]{2}\omega) \arrow[dl, dash] & \mathbb{Q}(\sqrt[3]{2}\omega^2) \arrow[dll, dash] \\
& & \mathbb{Q} & &
\end{tikzcd}$$

(b) $\mathbb{Q}(\sqrt[3]{2}, \omega)$ is the splitting field of x^3-2 over all of its subfields $\Rightarrow$ $\boxed{\text{all of them.}}$

(c) $K/\mathbb{Q}$ is normal $\Longleftrightarrow$ $Gal(\mathbb{Q}(\sqrt[3]{2}, \omega)/K)$ is a normal subgroup of $Gal(\mathbb{Q}(\sqrt[3]{2}, \omega), \mathbb{Q})$.

The subgroups which <u>aren't</u> normal are
$\langle \phi \rangle$, $\langle \theta\phi \rangle$, $\langle \theta^2\phi \rangle$

since $\theta(\phi)\theta^{-1} = \phi\,\theta^2(\theta^2) = \phi\theta. = \theta^2\phi.$

and $\theta(\theta\phi)\theta^{-1} = \theta\phi\theta^2\theta^2 = \theta\phi\theta = \phi.$

Hence, ~~all~~ every subfield of $\mathbb{Q}(\sqrt[3]{2}, \omega)$ is normal over $\mathbb{Q}$ <u>except</u> $\mathbb{Q}(\sqrt[3]{2})$, $\mathbb{Q}(\sqrt[3]{2}\omega)$, $\mathbb{Q}(\sqrt[3]{2}\omega^2)$.

$\boxed{8.3\#1}$ Note $\alpha^3\beta(\sqrt[4]{2}-i\sqrt[4]{2}) = \alpha^3(\sqrt[4]{2}+i\sqrt[4]{2})$

$$= \alpha^2(\sqrt[4]{2}i + i\sqrt[4]{2}i)$$

$$= \alpha(-\sqrt[4]{2} - i\sqrt[4]{2})$$

$$= (-\sqrt[4]{2}i + \sqrt[4]{2}),$$

So $\mathbb{Q}(\sqrt[4]{2}-i\sqrt[4]{2}) \subseteq \mathbb{Q}(\sqrt[4]{2}, i)^{\langle\alpha^3\beta\rangle}$.

But since $[\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}(\sqrt[4]{2}-i\sqrt[4]{2})] = |\langle\alpha^3\beta\rangle|$,

by the Fundamental Theorem,

$$\mathbb{Q}(\sqrt[4]{2}-i\sqrt[4]{2}) = \mathbb{Q}(\sqrt[4]{2}, i)^{\langle\alpha^3\beta\rangle}.$$

$\boxed{8.3\#2}$ Recall that conjugate subfields correspond to conjugate subgroups. Moreover, all subgroups of index 2 are normal, so we only have to consider the subgroups

$$\langle\beta\rangle, \langle\alpha^2\beta\rangle, \langle\alpha^2\rangle, \langle\alpha\beta\rangle, \langle\alpha^3\beta\rangle.$$

We also note that the center of G is

$\langle\alpha^2\rangle \Rightarrow \langle\alpha^2\rangle$ is normal.

Moreover, $\langle \alpha^2 \rangle \leq \mathrm{Stab}_G(g)$ for any $g \in G$.

So $\langle \beta, \alpha^2 \rangle \leq \mathrm{Stab}_G(\beta) \Rightarrow \beta$ has at most $\frac{|G|}{|\langle \beta, \alpha^2 \rangle|} = 2$

elt in its orbit (by Orbit-Stabilizer Thm).

Since $\alpha \beta \alpha^{-1} = \alpha^2 \beta$, we see that

$\langle \beta \rangle$ is conjugate to $\langle \alpha^2 \beta \rangle$ by α.

$\Rightarrow \mathbb{Q}(\sqrt[4]{2})$ is conj. to $\mathbb{Q}(i\sqrt[4]{2})$ by α.

Similarly, we can conclude that $\langle \alpha\beta \rangle$ is only conjugate to $\langle \alpha^3 \beta \rangle$ (and itself), where

$\beta (\alpha\beta) \beta^{-1} = \alpha^3 \beta$.

$\Rightarrow \mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2})$ is conj. to $\mathbb{Q}(\sqrt[4]{2} - i\sqrt[4]{2})$ by β.

$\boxed{8.3\ \#3}$ The roots of x^4+1 are $\frac{1}{\sqrt{2}}(\pm 1 \pm i)$, so the splitting field is $\mathbb{Q}(\sqrt{2}, i)$. The Galois group is $\mathbb{Z}_2 \times \mathbb{Z}_2$, generated by

$\alpha: \sqrt{2} \mapsto -\sqrt{2}$, $i \mapsto i$ $\qquad$ $\beta: \sqrt{2} \mapsto \sqrt{2}$, $i \mapsto -i$.

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$\boxed{8.3\#4}$ The roots of $x^4 - x^2 - 6 = (x^2-3)(x^2+2)$ are $\pm\sqrt{3}$ and $\pm i\sqrt{2}$. ~~So~~ So the S.F. is $\mathbb{Q}(\sqrt{3}, \sqrt{2}i)$. The Galois group is $\mathbb{Z}_2 \times \mathbb{Z}_2$, generated by

$$\alpha : \begin{array}{l} \sqrt{3} \mapsto -\sqrt{3} \\ i\sqrt{2} \mapsto i\sqrt{2} \end{array} \qquad \beta : \begin{array}{l} \sqrt{3} \mapsto \sqrt{3} \\ i\sqrt{2} \mapsto -i\sqrt{2}. \end{array}$$

$\boxed{8.3\#5}$ The roots of $x^8 - 1$ over $\mathbb{Q}$ are $\pm 1, \pm i, \frac{1}{\sqrt{2}}(\pm 1 \pm i)$. So the splitting field is $\mathbb{Q}(\sqrt{2}, i)$ as in Ex 8.3#3.

$\boxed{8.3\#6}$

$$\begin{array}{c} F \\ | \\ E \\ |\, 2 \\ K \end{array}$$

$[E:K] = 2 \Rightarrow [\mathrm{Gal}(F/K) : \mathrm{Gal}(F/E)] = 2$

$\Rightarrow \mathrm{Gal}(F/E) \trianglelefteq \mathrm{Gal}(F/K)$.

Thus E is normal over K.

$\boxed{8.3\#7}$ See, for example, Dr. PMH Wilson's notes on Galois Theory at

www.jchl.co.uk/maths/Galois.pdf.