HW #2

$\boxed{1}$ Because $[\mathbb{F}_{3^{16}} : \mathbb{F}_{3^2}] = 8$, Theorem 8.1.8 says that $Gal(\mathbb{F}_{3^{16}}/\mathbb{F}_{3^2}) \cong \mathbb{Z}_8$, generated by ϕ^2, where $\phi : \mathbb{F}_{3^{16}} \to \mathbb{F}_{3^{16}}$ by $x \mapsto x^3$.

$\boxed{2}$ The subgroups of $\mathbb{Z}_8$ correspond to divisors of 8: $\mathbb{Z}_8 = \langle \phi^2 \rangle$, $\mathbb{Z}_4 = \langle \phi^4 \rangle$, $\mathbb{Z}_2 = \langle \phi^8 \rangle$, $\{e\} = \langle \phi^0 \rangle$.

$\boxed{3}$ Claim: The fixed field of $\langle \phi^n \rangle$ is $\mathbb{F}_{3^n}$ for $n = 2, 4, 8, 16$.

Pf: $\phi^n(x) = x^{3^n}$, so every element of the fixed field satisfies $x^{3^n} - x = 0$. But the roots of this polynomial are precisely the elements of $\mathbb{F}_{3^4}$. //

Subgroups	Fixed fields
$\{e\}$	$\mathbb{F}_{3^{16}}$
$\vert$	$\vert$
$\mathbb{Z}_2$	$\mathbb{F}_{3^8}$
$\vert$	$\vert$
$\mathbb{Z}_4$	$\mathbb{F}_{3^4}$
$\vert$	$\vert$
$\mathbb{Z}_8$	$\mathbb{F}_{3^2}$

$\boxed{4}$ The roots of $x^3-2=0$ are $\sqrt[3]{2}\,\omega^i$,

where $\omega^3=1$. So clearly

$$\mathbb{Q}(\sqrt[3]{2},\omega) \supseteq \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{2}\omega, \sqrt[3]{2}\omega^2)$$
$$= \text{splitting field.}$$

We have the reverse inclusion because

$$\omega = \sqrt[3]{2}\omega \big/ \sqrt[3]{2}.$$

$\boxed{5}$ Looking at the (partial) field diagram tells us that

$$[\mathbb{Q}(\sqrt[3]{2},\omega):\mathbb{Q}] = 6.$$

$$\begin{tikzcd}
& \mathbb{Q}(\sqrt[3]{2},\omega) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\omega) \arrow[dr, dash, "2"'] & & \mathbb{Q}(\sqrt[3]{2}) \arrow[dl, dash, "3"] \\
& \mathbb{Q} &
\end{tikzcd} \quad \Big] \leq 6$$

Thus $G := \mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2},\omega)/\mathbb{Q})$ has order 6.

Define $\theta \in G$ by
$$\theta(\sqrt[3]{2}) = \sqrt[3]{2}\omega$$
$$\theta(\omega) = \omega.$$

$\phi \in G$ by
$$\phi(\sqrt[3]{2}) = \sqrt[3]{2}$$
$$\phi(\omega) = \omega^2$$

Note $\theta^3 = id$, since
$$\theta^3(\sqrt[3]{2}) = \theta^2(\sqrt[3]{2}\omega) = \theta(\sqrt[3]{2}\omega^2) = \sqrt[3]{2}\omega^3 = \sqrt[3]{2}$$
$$\theta^3(\omega) = \omega.$$

and $\phi^2 = id$ since
$$\phi^2(\sqrt[3]{2}) = \sqrt[3]{2}$$
$$\phi^2(\omega) = \phi(\omega^2) = \omega^4 = \omega.$$

Hence $\langle \theta, \phi \rangle \leqslant G$ has order at least 6

$\Rightarrow \langle \theta, \phi \rangle = G$.

Now $\phi \circ \theta(\sqrt[3]{2}) = \phi(\sqrt[3]{2}\omega) = \sqrt[3]{2}\omega^2$

$\theta \circ \phi(\sqrt[3]{2}) = \theta(\sqrt[3]{2}) = \sqrt[3]{2}\omega$.

So $\phi \circ \theta \neq \theta \circ \phi$, so G is not abelian.

The only nonabelian group of order 6 is S_3.

$\boxed{6}$ Any nontrivial proper subgroup of G has order 2 or 3, and hence is cyclic. Thus, we can list all of G's subgroups as follows:

$$\begin{tikzcd}
 & & G = \langle \theta, \phi \rangle \arrow[dll, no head] \arrow[d, no head] \arrow[dr, no head] \arrow[drr, no head] & & \\
\langle \theta \rangle = \langle \theta^2 \rangle \arrow[drr, no head, "3"'] & & \langle \phi \rangle \arrow[d, no head, "2"] & \langle \theta\phi \rangle \arrow[dl, no head, "2"'] & \langle \theta^2\phi \rangle \arrow[dll, no head, "2"] \\
 & & \{e\} & &
\end{tikzcd}$$

Note: ϕ, $\theta\phi$, $\theta^2\phi$ all have order 2. For example,

$(\theta\phi)^2(\sqrt[3]{2}) = \theta\phi\theta(\sqrt[3]{2}) = \theta\phi(\sqrt[3]{2}\omega) = \theta(\sqrt[3]{2}\omega^2) = (\sqrt[3]{2}\omega)\omega^2 = \sqrt[3]{2}$.

$(\theta\phi)^2(\omega) = \theta\phi\theta(\omega^2) = \theta\phi(\omega^2) = \theta(\omega^4) = \omega$.

$\boxed{7}$ Clearly $\mathbb{Q}(\omega) \subseteq \mathbb{Q}(\sqrt[3]{2}, \omega)^{\langle\theta\rangle} \subseteq \mathbb{Q}(\sqrt[3]{2}, \omega)$

index $|\langle\theta\rangle| = 3$

Since $[\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}(\omega)] = 3$, we know

$$\mathbb{Q}(\omega) = \mathbb{Q}(\sqrt[3]{2}, \omega)^{\langle\theta\rangle}.$$

Similarly, $\mathbb{Q}(\sqrt[3]{2}) = \mathbb{Q}(\sqrt[3]{2}, \omega)^{\langle\phi\rangle}$.

For $\langle\theta\phi\rangle$, we compute $\theta\circ\phi(\sqrt[3]{2}) = \theta(\sqrt[3]{2}) = \sqrt[3]{2}\,\omega$

$$\theta\circ\phi(\omega) = \theta(\omega^2) = \omega^2.$$

and we notice (or guess) that

$$\theta\circ\phi(\sqrt[3]{2}\,\omega^2) = \theta(\sqrt[3]{2}\,\omega) = \sqrt[3]{2}\,\omega^2.$$

so $\mathbb{Q}(\sqrt[3]{2}, \omega)^{\langle\theta\phi\rangle} = \mathbb{Q}(\sqrt[3]{2}\,\omega^2)$.

Similarly, $\mathbb{Q}(\sqrt[3]{2}, \omega)^{\langle\theta^2\phi\rangle} = \mathbb{Q}(\sqrt[3]{2}\,\omega)$.