MIDTERM SOLUTIONS

Chris Briggs

1. Compute the continued fraction of the following numbers:

   (a) (5 points) $\dfrac{1+\sqrt{5}}{2}$;

   (b) (5 points) $\sqrt{5}$.

   Solution: a)

   $\frac{1+\sqrt{5}}{2} = \frac{-1+\sqrt{5}}{2} + 1$. $a_0 = 1$

   $\frac{2}{-1+\sqrt{5}} = \frac{2(-1+\sqrt{5})}{4} = \frac{1+\sqrt{5}}{2}$. $a_1 = 1$, and

   $\frac{1+\sqrt{5}}{2}$ has been repeated. So

   $\frac{1+\sqrt{5}}{2} = [\bar{1}]$.

   b)

   $\sqrt{5} = -2 + \sqrt{5} + 2$. $a_0 = 2$.

   $\frac{1}{-2+\sqrt{5}} = \frac{2+\sqrt{5}}{1} = -2 + \sqrt{5} + 4$. $a_1 = 4$, and

   $-2 + \sqrt{5}$ has been repeated. So

   $\sqrt{5} = [2, \bar{4}]$.

2. Represent as real numbers the following continued fractions.

   (a) (5 points) $[1, \bar{3}]$;

   (b) (5 points) $[1, 2, 3]$;

   (c) (5 points) $[1, 2, \bar{3}]$.

   Solution: a)

   Let $x = [1, \bar{3}], y = [\bar{3}]$. $x = 1 + \frac{1}{y}$ and $y = 3 + \frac{1}{y}$,

   so $x - y = 2$. By a computation, $y^2 - 3y - 1 = 0$ so $y = \frac{3+\sqrt{13}}{2}$,

   and then $x = \frac{-1+\sqrt{13}}{2}$.

b)

$[1,2,3] = 1 + \frac{1}{2+\frac{1}{3}} = \frac{10}{3}$.

c)

$z = [1,2,\bar{3}] = 1 + \frac{1}{2+\frac{1}{y}}$, where y is as in 2(a).

So $z = 1 + \frac{1}{2+\frac{1}{y}} = 1 + \frac{1}{2+\frac{1}{\frac{3+\sqrt{13}}{2}}} = 1 + \frac{1}{2+\frac{2}{3+\sqrt{13}}} = 1 + \frac{1}{\frac{8+2\sqrt{13}}{3+\sqrt{13}}} = 1 + \frac{3+\sqrt{13}}{8+\sqrt{2}13} = \frac{11+3\sqrt{13}}{8+2\sqrt{13}} = \frac{5+\sqrt{13}}{6}$

3. (25 points) Prove that if p is an odd prime, then every reduced residue system modulo p contains exactly $\dfrac{p-1}{2}$ quadratic residues and $\dfrac{p-1}{2}$ quadratic nonresidues.

Solution:

Suppose S is the reduced system of residues $(mod p)$: $S = \{1, 2, \ldots, p-1\}$. We're interested in what conditions on x, y in S will cause them to have the same squares $(mod p)$. So choose $p-1 > x > y > 0$, and suppose $x^2 \equiv y^2 (mod p)$. Then $p|(x+y)(x-y)$, and by hypothesis on x, y, $0 < x-y < p$, hence $p|x+y$. But $x, y < p$ so $0 < x+y < 2p$, hence $x+y = p$. Conversely, if $x+y = p$ then $x^2 \equiv y^2 (mod p)$. So two elements of S have the same quadratic residue $(mod p)$ precisely when their sum is p. There are $|S|/2 = \frac{p-1}{2}$ such pairs.

4. (a) (10 points) Find the smallest positive integer solution, or prove that no such solution exist, to

$$x^2 - 5y^2 = -1.$$

(b) (10 points) Find the smallest positive integer solution, or prove that no such solution exist, to

$$x^2 - 5y^2 = 1.$$

(c) (20 points) Find all the positive integer solutions, or prove that no such solutions exist, to

$$x^2 - 5y^2 = 1.$$

Hint: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$.

(d) (10 points) Find *all* integer solutions to $x^2 - 5y^2 = 1$.

Solution:

a)

By calculation, $x = 1, y = 1$ fails, and $x = 2, y = 1$ works. So $(2, 1)$ is the smallest solution. A systematic approach would use the convergents of $\sqrt{5}$, the first of which is $\frac{2}{1}$.

b)

The smallest solution is gotten by squaring $2 + 1\sqrt{5}$, where $(2, 1)$ is taken from the solution in 4(a): $(2 + \sqrt{5})^2 = 9 + 4\sqrt{5}$, and we assign to x the rational part, to y the coefficient of $\sqrt{5}$. So $(9, 4)$ is the smallest positive solution.

c)

All solutions are derived by taking powers of the solution in 4(b). Specifically, solutions are the coefficients of 1 and $\sqrt{5}$ in $(9 + 4\sqrt{5})^n$, $n \geq 1$.
$(9+4\sqrt{5})^n = \sum_{k=0}^{n} \binom{n}{k} 9^{n-k}(4\sqrt{5})^k = \sum_{k_{even}} \binom{n}{k} 9^{n-k}(4\sqrt{5})^k + \sqrt{5}\sum_{k_{odd}} \binom{n}{k} 9^{n-k} 4^k \sqrt{5}^{k-1}$
$= \sum_{k_{even}} \binom{n}{k} 9^{n-k}(80)^{\frac{k}{2}} + 4\sqrt{5}\sum_{k_{odd}} \binom{n}{k} 9^{n-k}(80)^{\frac{k-1}{2}}$
So all positive solutions are given by $x_n = \sum_{k_{even}} \binom{n}{k} 9^{n-k}(80)^{\frac{k}{2}}$, $y_n = 4\sum_{k_{odd}} \binom{n}{k} 9^{n-k}(80)^{\frac{k-1}{2}}$

d)

(a, b) is a solution iff $(\epsilon_1 a, \epsilon_2 b)$ is a solution where $\epsilon_i \in \{\pm 1\}$, so in this way the positive solutions of 4(c) yield all integer solutions that contain no zeros. One must also include the "trivial" solution $x = 1, y = 0$ which is excluded by the method of 4(c).