HOMEWORK 6

DUE 28 FEBRUARY 2012

1. Determine whether 888 is a quadratic residue or nonresidue modulo the prime 1999 using exclusively the Legendre symbol.

2. Determine whether 888 is a quadratic residue or nonresidue modulo 1999 by factoring $888 = 2 \cdot 4 \cdot 111$ and using Jacobi symbols.

3. Same for -104 modulo the prime 997.

4. Use quadratic reciprocity to determine the congruence classes in $(\mathbb{Z}/84\mathbb{Z})^\times$ with $\left(\frac{-21}{p}\right) = 1$. This solves the reciprocity step when $n = 21$, i.e. it tells us when $p \mid a^2 + 21b^2$ for some relatively prime integers a, b.

The next two problems concern the homomorphism $\chi_D : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}$ of Theorem 8.17 from the notes. Recall that $D \equiv 0, 1 \pmod 4$ and that $\chi_D([m]) = \left(\frac{D}{m}\right)$ if m is an odd positive integer.

5. Show that

$$\chi_D([-1]) = \begin{cases} 1 & \text{if } D > 0; \\ -1 & \text{if } D < 0. \end{cases}$$

6. If $D \equiv 1 \pmod 4$, show that

$$\chi_D([2]) = \begin{cases} 1 & \text{if } D \equiv 1 \pmod 8; \\ -1 & \text{if } D \equiv 5 \pmod 8. \end{cases}$$

7. Extra credit We will show that Theorem 8.17 from the notes is equivalent to the quadratic reciprocity law (Theorem 8.10) and the supplementary laws (Corollary 8.3 and Proposition 8.6) for the Legendre symbol. We proved Theorem 8.17 using quadratic reciprocity. Now assume that Theorem 8.17 holds for all nonzero integers $D \equiv 0, 1 \pmod 4$.

(a) Let p, q be distinct odd primes and set $q^* = (-1)^{\frac{q-1}{2}} q$. By applying Theorem 8.17 with $D = q^*$ (you first have to show that you *can* apply it in this case!), show that $\left(\frac{q^*}{\cdot}\right)$ induces a homomorphism $(\mathbb{Z}/q\mathbb{Z})^\times \to \{\pm 1\}$.

(b) Since $\left(\frac{\cdot}{q}\right)$ can also be regarded as a homomorphism $(\mathbb{Z}/q\mathbb{Z})^\times \to \{\pm 1\}$, and $(\mathbb{Z}/q\mathbb{Z})^\times$ is cyclic, show that

$$\left(\frac{q^*}{\cdot}\right) = \left(\frac{\cdot}{q}\right).$$

(c) Use similar arguments to prove Corollary 8.3 and Proposition 8.6.
Hint: Apply Theorem 8.17 for $D = -4$ and $D = 8$ respectively.