HOMEWORK 3

DUE 7 FEBRUARY 2012

1. Let $A = [a_0, a_1, a_2, \ldots,]$. For each $n \geq 0$ set

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$$

where we treat $a_0, a_1, \ldots$ as variables, rather than as specific numbers. That is, you don't have to worry about simplifying a fraction of form $\frac{a_0+a_1}{a_2}$ even though for some choice of numbers a_i the numerator and denominator might have common factors. But $\frac{10a_0a_2+10a_1a_2}{15a_2^2} = \frac{2a_0+2a_1}{3a_2}$. Show that

$$p_0 = a_0, \qquad p_1 = a_0a_1 + 1, \quad \text{and} \quad p_{n+1} = a_{n+1}p_n + p_{n-1} \text{ for } n \geq 1;$$

$$q_0 = 1, \qquad q_1 = a_1, \quad \text{and} \quad q_{n+1} = a_{n+1}q_n + q_{n-1} \text{ for } n \geq 1.$$

2. Extra credit Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers with $a_n \geq 1$ for each $n \geq 0$. Set

$$u_n = [a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

Prove that the limit $\lim_{n\to\infty} u_n$ exists. (*Hint:* Use the previous exercise and the result proved in class about the differences of successive convergents to show that $u_0, u_1, \ldots$ is a Cauchy sequence.)

3. Show that a real number A is rational if and only if its continued fraction expansion is finite.

4. Find the value of each of the following periodic continued fractions. Express your answer in the form

$$\frac{a + b\sqrt{d}}{c},$$

where a, b, c, d are integers.

(a) $[\overline{1}] = [1, 1, 1, \ldots]$

(b) $[\overline{1, 2, 3}] = [1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots]$

(c) $[1, \overline{2, 3}] = [1, 2, 3, 2, 3, 2, 3, \ldots]$

5. For each of the following numbers find their (periodic) continued fraction. What's the period in each case?

(a) $\dfrac{16 - \sqrt{3}}{11}$

(b) $\dfrac{1 + 2\sqrt{5}}{3}$

(c) $\dfrac{1+\sqrt{5}}{2}$.

6. If p is a prime, show that Pell's equation

$$x^2 - py^2 = -1$$

has integer solutions if and only if $p = 2$ or $p \equiv 1 \pmod 4$.
Hint: Consider $a+1$ and $a-1$ where (a,b) is a solution to $a^2 - pb^2 = 1$.

7. Find the smallest positive integer solution to the following equations or show that no positive integer solutions exist.

(a) $x^2 - 31y^2 = 1$

(b) $x^2 - 30y^2 = -1$

(c) $x^2 - 29y^2 = 1$.