MATH 20C Lectures 25 & 26
Monday, November 22 & Wednesday, November 24 2010

taught by J. Lebl

Here is a list of the examples explained by Prof. Lebl. I am not going to write down the solutions, since I think it would be a good exercise for everyone to go through them and compute on their own. The one exception is Example 3, because this is a novel way of using double integrals. The multivariable techniques are used to compute a single variable integral which could not be computed with only single variable calculus knowledge. For an exposition of polar and cylindrical coordinates, see the notes for week 10.

Example 1 Integrate $xy+y^2$ over the region in plane described in polar coordinates by $1 \le r \le 2$, $-\pi/2 \le \theta \le \pi/2$.

This is a half annulus. In polar coordinates, $xy + y^2 = r^2 \cos\theta \sin\theta + r^2 \sin^2\theta$.

Example 2 $\iint_D xy dA$, where $D : x \ge 1, y \ge 0, (x-1)^2 + y^2 \le 1$.

Example 3 $\int_{-\infty}^{\infty} e^{-x^2} dx$.

Denote by A our integral. It will be non-negative since the exponential is positive. Then

$$A^2 = A \cdot A = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dx\, dy.$$

Changing to polar coordinates, this gives $A^2 = \displaystyle\int_0^{2\pi}\int_0^{\infty} re^{-r^2} dr$.

The inner integral is equal, via the change of variables $u = r^2$, to

$$\frac{1}{2}\int_0^{\infty} e^{-u} du = \frac{1}{2}.$$

Hence $A^2 = \pi$, and $A = \sqrt{\pi}$.

Example 4 Let $W : x^2 + y^2 \le 1, 0 \le z \le 1 + x^2 + y^2$.

$$\iiint_W (x^2 + y^2 - z)\, dV = \int_0^{2\pi}\int_0^1\int_0^{1+r^2} (r^2 - z) r dz dr d\theta = \ldots$$

Example 5 The volume of the unit ball $\mathbb{B}$ in $\mathbb{R}^3$ can be computed using cylindrical coordinates ($4\pi/3$).