MATH 20C Lecture 22 - Monday, November 15, 2010

Double integrals

Recall integral in 1-variable calculus: $\int_a^b f(x)dx =$ area below graph $y = f(x)$ over $[a, b]$.

Now: double integral $\displaystyle\iint_R f(x, y)dA =$ volume below graph $z = f(x, y)$ over region R in the xy-plane.

Cut R into small pieces $\Delta A_i \implies$ the volume is approximately $\sum f(x_i, y_i)\Delta A_i$. Limit as $\Delta A \to 0$ gives $\iint f(x, y)dA$. (demo: potato cut into french fries)

How to compute the integral? By taking slices: $S(x) =$ area of the slice by a plane parallel to yz-plane (demo: potato chips); then

$$\text{volume } = \int_{x_{\min}}^{x_{\max}} S(x)dx \quad \text{and for given } x, \quad S(x) = \int_{y_{\min}(x)}^{y_{\max}(x)} f(x, y)dy$$

BEWARE! The limits of integration in y depend on x!

In the inner integral, x is a fixed parameter, y is the integration variable. We get an *iterated integral.*

Example 1 $f(x, y) = 1 - x^2 - y^2$ and $R : 0 \le x \le 1, 0 \le y \le 1$. (Shown computer graphics.)

$$\int_0^1 \int_0^1 \left(1 - x^2 - y^2\right) dy\, dx$$

How to evaluate?

1) inner integral (x is constant):

$$\int_0^1 \left(1 - x^2 - y^2\right) dy = \left[y - x^2 y - \frac{y^3}{3}\right]_{y=0}^{y=1} = \left(1 - x^2 - \frac{1}{3}\right) - 0 = \frac{2}{3} - x^2.$$

2)outer integral: $\displaystyle\int_0^1 \left(\frac{2}{3} - x^2\right) dx = \left[\frac{2}{3}x - \frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$

Note: $dA = dy\, dx = dx\, dy$, limit of $\Delta A = \Delta y \Delta x = \Delta x \Delta y$ for small rectangles.

Example 2 Same function over the quarter-disk $R : x^2 + y^2 \le 1, 0 \le x \le 1, 0 \le y \le 1$. (computes volume between xy-plane and paraboloid in the first octant).

How to find the bounds of integration? Fix x constant and look at the slice of R parallel to y-axis. Bounds from $y = 0$ to $y = \sqrt{1 - x^2}$ in the inner integral. For the outer integral: first slice is at $x = 0$, last slice is at $x = 1$. So we get

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \left(1 - x^2 - y^2\right) dy\, dx.$$

Note that the inner bounds depend on the outer variable x; the outer bounds are constants!

1) inner integral (x is constant):

$$\int_0^{\sqrt{1-x^2}} \left(1 - x^2 - y^2\right) dy = \left[(1 - x^2)y - \frac{y^3}{3}\right]_{y=0}^{y=\sqrt{1-x^2}} = (1 - x^2)^{3/2} - \frac{(1 - x^2)^{3/2}}{3} = \frac{1}{3}(1 - x^2)^{3/2}.$$

2)outer integral:

$$\int_0^1 \frac{1}{3}(1-x^2)^{3/2}dx = \ldots (\text{trig substitution } x = \sin\theta, \text{ double angle formulas}) \ldots = \frac{\pi}{8}.$$

This is complicated! It will be easier to do it in polar coordinates.

MATH 20C Lecture 23 - Wednesday, November 17, 2010

Exchanging the order of integration

$\int_0^1\int_0^2 f(x,y)dx\,dy = \int_0^2\int_0^1 f(x,y)dy\,dx$, since region is a rectangle (drawn picture). In general, more complicated!

Example: $\displaystyle\int_0^1\int_x^{\sqrt{x}} \frac{e^y}{y}dy\,dx$ (Inner integral has no formula.)

To exchange order: 1) draw the region (here: $x \le y \le \sqrt{x}$ for $0 \le x \le 1$ – picture drawn on blackboard).

2) figure out bounds in other direction: fixing a value of y, what are the bounds for x? H: left border is $x = y^2$, right is $x = y$; first slice is $y = 0$, last slice is $y = 1$, so we get

$$\int_0^1\int_{y^2}^{\sqrt{y}} \frac{e^y}{y}dx\,dy = \int_0^1 \frac{e^y}{y}(y-y^2)dy = \int_0^1 e^y(1-y)dy \overset{\text{parts}}{=} [e^y(1-y)]_{y=0}^{y=1} + \int_0^1 e^y dy = e-2.$$

Applications

Computing volumes *Example:* Problem 55, section 15.2. Find the volume of the region enclosed by $z = 1-y^2$ and $z = y^2-1$ for $0 \le x \le 2$.

Both surfaces look like parabola-shaped tunnels along the x-axis. They intersect at $1-y^2 = y^2-1 \implies y = \pm 1$. So $z = 0$ and x can be anything, therefore lines parallel to the x-axis. I tried to draw a picture, but I got it rotated by 90°. To be corrected next time. Get volume by integrating the difference $z_{\text{top}} - z_{\text{bottom}}$, i.e. take the volume under the top surface and subtract the volume under the bottom surface (same idea as in 1 variable).

$$\begin{aligned}\pm\text{ vol } &= \int_0^2\int_{-1}^1 \big((1-y^2)-(y^2-1)\big)\,dy\,dx = 2\int_0^2\int_{-1}^1(1-y^2)dy\,dx \\ &= 2\int_0^2\left[y-\frac{y^3}{3}\right]_{y=-1}^{y=1}dx = 2\int_0^2 \frac{4}{3}dx = \frac{16}{3}.\end{aligned}$$

Since volume is always positive, our answer is 16/3.

Area of a plane region R is

$$\text{area}(R) = \iint_R 1 dA.$$

Mass the total mass of a flat object in the shape of a region R with density given by $\rho(x,y)$ is

$$\text{Mass} = \iint_R \rho(x,y)dA.$$

Average the average value of a function $f(x,y)$ over the plane region R is

$$\bar{f} = \frac{1}{\text{area}(R)} = \iint_R f(x,y)dA.$$

Weighted average of the function $f(x,y)$ over the plane region R with density $\rho(x,y)$ is

$$\frac{1}{\text{Mass}} \iint_R f(x,y)\rho(x,y)dA.$$

Center of mass of a plate with density $\rho(x,y)$ is the point with coordinates $(\bar{x},\bar{y})$ given by weighted average

$$\bar{x} = \frac{1}{\text{Mass}} \iint_R x\rho(x,y)dA,$$
$$\bar{y} = \frac{1}{\text{Mass}} \iint_R y\rho(x,y)dA.$$

Example: Problem 57, section 15.2. A plate in the shape of the region bounded by $y = x^{-1}$ and $y = 0$ for $1 \leq xleq4$ has mass density $\rho(x,y) = y/x$. Calculate the total mass of the plate.

First, draw region. Then set limits of integration.

$$\text{Mass} = \int_0^4 \int_0^{x^{-1}} \frac{y}{x} dy\, dx = \int_0^4 \left[\frac{y^2}{2x}\right]_{y=0}^{y=x^{-1}} dx = \frac{1}{2}\int_0^4 x^{-3}dx = -\frac{1}{4}\left[\frac{1}{x^2}\right]_{x=0}^{x=4} = \frac{15}{64}.$$

MATH 20C Lecture 24 - Friday, November 19, 2010

I made the correction for the drawing of the region in Problem 55 that was discussed last time.

Triple Integrals

$$\iiint_R f(x,y,z)\, dV \quad (R \text{ is a solid in space})$$

Note: $\Delta V = \text{area(base)} \cdot \text{height} = \Delta A \Delta z$, so $dV = dA\, dz = dx\, dy\, dz$ or any permutation of the three.

Example 1 $R:$ the region between paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$. (picture drawn)

The volume of this region is $\iiint_R 1\, dV = \iint_D \left[\int_{x^2+y^2}^{4-x^2-y^2} dz\right] dA$, where D is the shadow in the xy-plane of the region R.

To set up bounds, (1) for fixed (x,y) find bounds for z : here lower limit is $z = x^2 + y^2$, upper limit is $z = 4 - x^2 - y^2$; (2) find the shadow of R onto the xy-plane, i.e. set of values of

(x, y) above which region lies. Here: R is widest at intersection of paraboloids, which is in plane $z = 2$; general method: for which (x, y) is z on top surface $\geq z$ on bottom surface? Answer: when $4 - x^2 - y^2 \geq x^2 + y^2$, i.e. $x^2 + y^2 \leq 2$. So we integrate over a disk of radius $\sqrt{2}$ in the xy-plane. By usual method to set up double integrals, we finally get

$$\text{vol}(R) = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} dz\, dy\, dx.$$

Actual evaluation would be easier using polar coordinates.

Example 2 (based on Example 4 from Section 15.3 in the book) Let R be the region in the first octant (i.e. $x \leq 0, y \leq 0, z \leq 0$) bounded by

$$z = 4 - y^2, \quad y = 2x, \quad z = 0, \quad x = 0.$$

Consider $f(x, y, z) = xyz$. Set up the integral $\iiint_R f(x, y, z)\, dV$, in two different ways.

Showed picture of R and its projections on the various coordinate planes.

Setup #1 Integrate first with respect to z, then with respect to y.

The surface $z = 4 - y^2$ intersects the first quadrant of the xy-plane in the line $y = 2$. The projection of the xy-plane is a triangle bounded by the y-axis and the lines $y = 2$ and $y = 2x$. For each point (x, y) the vertical segment above it goes from $z = 0$ to $z = 4 - y^2$. Get

$$\int_0^1 \int_{2x}^2 \int_0^{4-y^2} xyz\, dz\, dy\, dx.$$

Setup #2 Integrate first with respect to x, then with respect to z.

The projection onto the yz-plane is bounded by the y-axis, the z-axis and the parabola $z = 4-y^2$. For each point (y, z) the segment in the direction of the x-axis goes from $x = 0$ to $x = y/2$. Get

$$\int_0^2 \int_0^{4-y^2} \int_0^{y/2} xyz\, dx\, dz\, dy.$$

Applications

Mass the total mass of a solid R with density given by $\rho(x, y, z)$ is

$$\text{Mass (R)} = \iiint_R \rho(x, y, z)\, dV.$$

Average the average value of a function $f(x, y, z)$ over the a solid R is

$$\bar{f} = \frac{1}{\text{vol}(R)} = \iiint_R f(x, y, z)\, dV.$$

Weighted average of the function $f(x, y, z)$ over the solid R with density $\rho(x, y, z)$ is

$$\frac{1}{\text{Mass}(R)} \iiint_R f(x, y, z)\rho(x, y, z)\, dV.$$

Center of mass of a solid with density $\rho(x,y,z)$ is the point with coordinates $(\bar{x},\bar{y},\bar{z})$ given by weighted average

$$\bar{x} = \frac{1}{\text{Mass}(R)} \iiint_R x\rho(x,y,z)\,dV,$$
$$\bar{y} = \frac{1}{\text{Mass}(R)} \iiint_R y\rho(x,y,z)\,dV,$$
$$\bar{z} = \frac{1}{\text{Mass}(R)} \iiint_R z\rho(x,y,z)\,dV.$$

Example: Did example 6, section 15.3. Let R be a solid in the shape of the first octant of the unit ball. Assume the density is given by $\rho(x,y,z) = y$. Find the z-coordinate of the center of mass of R. *Solution* First drawn picture of R. The unit sphere has equation $x^2+y^2+z^2=1$. It intersects the xy-plane in the unit circle $x^2+y^2=1$. We want only the parts with $x \le 0, y \le 0, z \le 0$.

$$\begin{aligned}\bar{z} &= \iiint_R z\rho(x,y,z)\,dV = \iint_{\text{quarter unit disk}} \left[\int_0^{\sqrt{1-x^2-y^2}} yz\,dz\right] dA \\ &= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} yz\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{1-x^2}} \left[y\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{1-x^2-y^2}} dy\,dx = \ldots = \frac{1}{15}.\end{aligned}$$