MATH 20C Lecture 13 - Monday, October 25, 2010

Recall chain rule I: $g = F(u)$ and $u = u(x, y)$, then $\frac{\partial g}{\partial x} = \frac{dF}{du}\frac{\partial u}{\partial x}$. Used this to compute the partial derivatives of $g(x, y, z) = \ln(x^2 + y^2 - xz)$. Get

$$\frac{\partial g}{\partial x} = \frac{2x - z}{x^2 + y^2 - xz}, \quad \frac{\partial g}{\partial y} = \frac{2y}{x^2 + y^2 - xz}, \quad \frac{\partial g}{\partial z} = \frac{-x}{x^2 + y^2 - xz}.$$

Higher order partial derivatives

Are computed by taking successive partial derivatives. For instance $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ and so on.

Computed

$$\frac{\partial^2 g}{\partial z \partial x} = \frac{\partial x}{\partial z}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial z}\left(\frac{2x - z}{x^2 + y^2 - xz}\right) = \frac{(-1)(x^2 + y^2 - xz) - (2x - z)(-x)}{(x^2 + y^2 - xz)^2}$$

$$\frac{\partial^2 g}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial z}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{x^2 + y^2 - xz}\right) = \frac{(-1)(x^2 + y^2 - xz) - (-x)(2x - z)}{(x^2 + y^2 - xz)^2}$$

Notice that $\frac{\partial^2 g}{\partial z \partial x} = \frac{\partial^2 g}{\partial x \partial z}$. This is no coincidence. In general,

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f}{\partial y \partial x}}$$

MATH 20C Lecture 14 - Wednesday, October 27, 2010

Recall that the gradient vector of $f(x, y, z)$ is $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$. Using this notation, the chain rule can be re-written as follows. Consider a function $f(x, y, z)$ with $x = x(t), y = y(t), z = z(t)$. On the path described by $\vec{r}(t) = \langle x(t), y(t) \rangle$, we have

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} = \nabla f \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle.$$

That is,

$$\boxed{\frac{df}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt} = \nabla f \cdot \vec{v}}$$

where $\vec{v}$ is the velocity vector.
Note: ∇f is a vector whose value depends on the point (x, y) where we evaluate f.
Theorem: ∇f is perpendicular to the level surfaces $f = c$. Proof: take a curve $\vec{r} = \vec{r}(t)$ contained inside level surface $f = c$. Then velocity $\vec{v} = d\vec{r}/dt$ is in the tangent plane, and by chain rule, $dw/dt = \nabla f \cdot vecv = 0$, so $\vec{v} \perp \nabla f$. This is true for every $\vec{v}$ in the tangent plane.
Example: $f = x^2 + y^2$, then $f = c$ are circles, $\nabla w = \langle 2x, 2y \rangle$ points radially out so $\perp$ circles.

Application: the tangent plane to a surface $f(x, y, z) = c$ at a point P is the plane through P with normal vector $\nabla f(P)$.

Example: tangent plane to $x^2 + y^2 - z^2 = 4$ at $(2, 1, 1)$: gradient is $\langle 2x, 2y, -2z\rangle = \langle 4, 2, -2\rangle$; tangent plane is $4x + 2y - 2z = 8$. (Here we could also solve for $z = \pm\sqrt{x^2 + y^2 - 4}$ and use linear approximation formula, but in general we can't.)

Another way to get the tangent plane: $\Delta f \approx 4\Delta x + 2\Delta y - 2\Delta z$. On the level surface we have $\Delta f = 0$, so its tangent plane approximation is $4\Delta x+2\Delta y-2\Delta z = 0$, i.e. $4(x-2)+2(y-1)-2(z-1) = 0$, same as above.

Example: Find the equation of the tangent line at the point $P = (1, 0, 1)$ to the curve obtained by intersecting the surfaces $x^2 + y^2 + z^2 = 2$ and $x^2 + y^2 - z^3 = 0$.

One could try to parametrize the curve and then find the tangent line, but that's hard. Instead, set $f = x^2 + y^2 + z^2$ and $g = x^2 + y^2 - z^3$. The tangent line is the intersection of the tangent plane to the first surface $f = 2$ with the tangent plane to the second surface $g = 0$. The two planes have normal vectors $\nabla f(P) = \langle 2, 0, 2\rangle$ and $\nabla g(P) = \langle 2, 0, -3\rangle$ respectively. Both these vectors are therefore $\perp$ to the tangent line, which means that the tangent line $\parallel$ to their cross product. Since $\nabla f(P) \times \nabla g(P) = \langle 0, 10, 0\rangle$, the equation of the tangent line is

$$\vec{L}(\theta) = \langle 1, 0, 1\rangle + \theta\langle 0, 10, 0\rangle.$$

Directional derivatives

We want to know the rate of change of f as we move (x, y) in an arbitrary direction.

Take a unit vector $\hat{u}$ and look at the cross-section of the graph of f by the vertical plane parallel to $\hat{u}$ and passing through the point (x, y). This is a curve passing through the point $P = (x, y, z = f(x, y))$ and we want to compute the slope the tangent line to this curve at P.

Notice that $\frac{\partial f}{\partial x}$ is the directional derivative in the direction of $\hat{\imath}$ and $\frac{\partial f}{\partial y}$ is the directional derivative in the direction of $\hat{\jmath}$.

MATH 20C Lecture 15 - Friday, October 29, 2010

Directional derivatives

Notation: $D_{\hat{u}} f(x_0, y_0)$ denotes the derivative of f in the direction of the unit vector $\hat{u}$ at the point (x_0, y_0).

Shown $f = x^2 + y^2 + 1$, and rotating slices through a point of the graph.

How to compute

Say that $\hat{u} = \langle a, b\rangle$. In order to compute $D_{\hat{u}} f(x_0, y_0)$, look at the straight line trajectory $\vec{r}(s)$ through (x_0, y_0) with velocity $\hat{u}$ given by $x(s) = x_0+as$, $y(s) = y_0+bs$. Then by definition $D_{\hat{u}} f(x_0, y_0) = \dfrac{df}{ds}$. This we can compute by chain rule to be $\dfrac{df}{ds} = \nabla f \cdot \dfrac{d\vec{r}}{ds}$. Hence

$$\boxed{D_{\hat{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \hat{u}.}$$

Example 1 Compute the directional derivative of $f = x^2 + y^3$ at $P = (2, 1)$ in the direction of $\vec{v} = \langle 4, 3 \rangle$.

$\nabla f = \langle 2x, 3y^2 \rangle$ so $\nabla f(P) = \langle 4, 3 \rangle$. The unit vector in the direction of $\vec{v}$ is $\hat{u} = \vec{v}/|\vec{v}| = \langle 4/5, 3/5 \rangle$. So $D_{\hat{u}} f(P) = \nabla f(P) \cdot \hat{u} = 5$. Therefore f is increasing in the direction of $\vec{v}$.

Example 2 Compute the directional derivative of $g = xe^{-yz}$ at $P = (1, 2, 0)$ in the direction of $\vec{v} = \langle 1, 1, 1 \rangle$.

$\nabla g = \langle e^{-yz}, -xze^{-yz}, -xye^{-yz} \rangle$ so $\nabla g(P) = \langle 1, 0, -2 \rangle$. The unit vector in the direction of $\vec{v}$ is $\hat{u} = \vec{v}/|\vec{v}| = \langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3} \rangle$. So $D_{\hat{u}} g(P) = \nabla g(P) \cdot \hat{u} = -1/\sqrt{3}$. Therefore g is decreasing in the direction of $\vec{v}$.

Geometric interpretation: $D_{\hat{u}} f = \nabla f \cdot \hat{u} = |\nabla f| cos\theta$. Maximal for $\cos\theta = 1$, when $\hat{u}$ is in direction of ∇f. Hence: direction of ∇f is that of fastest increase of f, and $|\nabla f|$ is the directional derivative in that direction.
It's minimal in the opposite direction.
We have $D_{\hat{u}} f = 0$ when $\hat{u} \perp \nabla f$, i.e. when $\hat{u}$ is tangent to direction of level surface.

Chain rule with more variables

For example $w = f(x, y)$, $x = x(u, v), y = y(u, v)$. Then we can view f as a function of u and v. The partial derivatives with respect to these new variables are

$$\boxed{\begin{aligned} \frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \end{aligned}}$$

The idea behind each formula is that changing u causes both x and y to change, at rates $\partial x/\partial u$ and $\partial y/\partial u$. The change in x affects f at the rate of $\partial f/\partial x$, for a total effect of $\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}$. At the same time, the change in y affects f at the rate of $\partial f/\partial y$, for a total effect of $\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$. Finally, the two effects add up to produce the change in f given by the first line in the boxed formula.

Example: polar coordinates.
$x = r\cos\theta, y = r\sin\theta$. Then $\dfrac{df}{dr} = f_x \dfrac{\partial x}{\partial r} + f_y \dfrac{\partial y}{\partial r} = f_x \cos\theta + f_y \sin\theta$, and similarly $\dfrac{df}{d\theta}$.