MATH 20C Lecture 27 - Monday, November 29, 2010

Polar coordinates

Recall: in the plane, $x = r\cos\theta, y = r\sin\theta$ where r is the distance from the origin to the (x, y) point, θ is the angle with the positive x-axis. Drawn picture.
Useful if either integrand or region have a simpler expression in polar coordinates.
Area element: $\Delta A \approx (r\Delta\theta)\Delta r$ (picture drawn of a small element with sides Δr and $r\Delta\theta$). Taking $Deltar, \Delta\theta \to 0$, we get

$$\boxed{dA = r\,dr\,d\theta.}$$

Example (from way back in Lecture 22):

$$\iint_{x^2+y^2\le 1, 0\le x\le 1, 0\le y\le 1} \left(1 - x^2 - y^2\right) dxdy = \int_0^{\pi/2}\int_0^1 (1-r^2)r\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{r=1} d\theta = \frac{\pi}{8}.$$

Once again,

$$\boxed{\iint_R f(x,y)dA = \iint_R f(r,\theta)r\,dr\,d\theta.}$$

In general: when setting up $\iint f r\,dr\,d\theta$, find bounds as usual: given a fixed θ, find initial and final values of r (sweep region by rays).

Cylindrical coordinates

(r, θ, z) where $x = r\cos\theta, y = r\sin\theta$. (Drawn picture.) Here r measures distance from z-axis, θ measures angle from xz-plane, z is still the height.
Cylinder of radius 8 centered on z-axis is $r = 8$ (drawn); $\theta = \pi/3$ is a vertical half-plane (drawn).
Volume element: $dV = dAdz$; in cylindrical coordinates, $dA = r\,dr\,d\theta$, so

$$\boxed{dV = r\,dr\,d\theta\,dz.}$$

Example (from Lecture 23): R : the region between paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$. The shadow on the xy-plane is the disk $x^2 + y^2 \le 2$ of radius $\sqrt{2}$. The volume of R is

$$\iiint_R 1\,dV = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2} (4 - r^2) r\,dz\,dr\,d\theta = \dots$$

Once again,

$$\boxed{\iiint_R f(x,y,z)dV = \iiint_R f(r,\theta,z)r\,dr\,d\theta\,dz.}$$

Example R : portion of the half-cylinder $x^2 + y^2 \le 4$, $x \ge 0$ such that $0 \le z \le 3y$. Compute the mass of the solid in the shape of R with mass-density given by $\rho(x, y, z) = z^2$.

Again, it's natural to set this up in cylindrical coordinates. The bounds for z are clear: $z_{\text{min}} = 0$ and $z_{\text{max}} = 3y = 3r\sin\theta$. The shadow on the xy-plane is the quarter disk $x^2 + y^2 \le 1, x \le 0, y \le 0$.

$$\text{Mass}(R) = \int_0^{\pi/2}\int_0^1\int_0^{3r\sin\theta} z^2 r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r\left[\frac{z^3}{3}\right]_{z=0}^{z=3r\sin\theta} dr\,d\theta$$
$$= \int_0^{\pi/2}\int_0^1 9r^4\sin^3\theta\,dr\,d\theta = 9\int_0^{\pi/2}\frac{32}{5}\sin^3\theta\,d\theta.$$

To evaluate this last integral, write $\sin^3\theta = \sin\theta(1-\cos^2\theta)$ and use the substitution $u = \cos\theta$. Do not forget to change the bounds of integration!

MATH 20C Lecture 28 - Wednesday, December 1, 2010

Review topics

- vectors: quantities that have length and direction
- operation with vectors (addition, subtraction, multiplication by scalars, dot product, cross product);
- determinants, area, volume;
- equations of lines and planes in 3-space (normal vectors)
- decomposing a vector into a components along specified direction (projection, $\vec{v}_{\parallel}$ and $\vec{v}_{\perp}$)
- parametric equations
- calculus with vectors: limits, differentiation (product rules, chain rule), integration;
- velocity and acceleration vectors; speed and arc length
- tangent line to a trajectory
- Functions of several variables, contour plots.
- Partial derivatives, gradient; approximation formulas, tangent planes, directional derivatives.
- higher order partial derivatives
- chain rule, change of variables, implicit differentiation
- Min/max problems: critical points, second derivative test, checking boundary. (least squares won't be on the exam)
- Min/max for non-independent variables: Lagrange multipliers. *Second derivative test does NOT work! Plug in values or use geometry.*

MATH 20C Lecture 29 - Friday, December 3, 2010

Review (continued)

- Double integrals: drawing picture of region, taking slices to set up the iterated integral
- Same in polar coordinates (recall that $dA = dx\,dy = r\,dr\,d\theta$).

  Example: Write in polar coordinates $\iint_R f(x,y)dA$ for R the disk of radius 1 centered at $(1,0)$. (Picture drawn)

  Since we are at the right of the y-axis, get $-\pi/2 \leq \theta \leq \pi/2$. For each, theta can use geometry to see $0 \leq r \leq 2\cos\theta$. Or can deduce the same with algebra, as follows. The equation of the circle is $(x-1)^2 + y^2 = 1$. Substitute polar coordinates ($x = r\cos\theta, y = r\sin\theta$) and get $r(r - 2\cos\theta) = 0$. Hence $r = 0$ and $r = 2\cos\theta$ are the two endpoints of the ray at θ. Get

  $$\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} f(r,\theta) r\,dr\,d\theta.$$

- Triple integrals in rectangular and cylindrical coordinates ($dV = dA\,dz$): setup, pictures.
- For evaluation, need to know: usual basic integrals (e.g. $\int \frac{dx}{x}$); integration by substitution (e.g. $\displaystyle\int_0^1 \frac{2tdt}{\sqrt{1+t^2}} = \int_1^2 \frac{du}{\sqrt{u}}$ by setting $u = 1 + t^2$), integration by parts. DO NOT need to know: complicated trigonometric integrals (e.g. $\int \cos^4\theta d\theta$).
- Applications: area, volume, mass, average and weighted average of a function, center of mass.

Next we discussed problems 28, 30 and 31 from the study guide.

This is it. Good luck on the exam!