Name: _SOLUTIONS — yellow version_

PID: _____

TA: _____

Section #: _____ Section time: _____

There are 7 pages and 4 questions, for a total of 100 points.

No notes, no calculators, no books.

Please turn off all electronic devices.

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. Unless otherwise stated, show all your work for full credit. To maximize credit, cross out incorrect work.

Good luck! ☺

Question:	1	2	3	4	Total
Points:	15	30	25	30	100
Score:					

1. (15 points) The variables x, y, z satisfy the relation $xyz^3 + z = 10$. Find $\dfrac{\partial z}{\partial y}$ at the point $(1, 1, 2)$.

$$xyz^3 + z = 10$$

Take $\dfrac{\partial}{\partial y}$ of both sides of the equation.

$$x\left[1 \cdot z^3 + y\left(3z^2 \frac{\partial z}{\partial y} \right) \right] + \frac{\partial z}{\partial y} = 0$$

Plug in $x = 1$ $y = 1$ $z = 2$

$$1\left[8 + 12 \frac{\partial z}{\partial y} \right] + \frac{\partial z}{\partial y} = 0$$

$$8 + 13 \frac{\partial z}{\partial y} = 0$$

$$\boxed{\frac{\partial z}{\partial y} = -\frac{8}{13}}$$

2. Let $f(x, y, z) = ye^{x^2 - z^2}$.

    (a) (5 points) Find ∇f at $(2, 1, 2)$.

$$\nabla f = \left\langle 2xy\, e^{x^2 - z^2},\ e^{x^2 - z^2},\ -2yz\, e^{x^2 - z^2} \right\rangle$$

$$\nabla f(2,1,2) = \left\langle 4e^0,\ e^0,\ -4e^0 \right\rangle$$

$$= \boxed{\langle 4, 1, -4 \rangle}$$

    (b) (5 points) Write the equation for the tangent plane to the surface $f = 1$ at the point $(2, 1, 2)$.

normal vector: $\langle 4, 1, -4 \rangle = \nabla f(2,1,2)$

$$4x + y - 4z = 4 \cdot 2 + 1 - 4(2)$$

$$\boxed{4x + y - 4z = 1}$$

(c) (10 points) Use a linear approximation to find the approximate value of $f(1.9, 1.1, 2)$.

$$\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z$$

$$f(1.9, 1.1, 2) - f(2,1,2) \approx f_x (1.9-2) + f_y(1.1-1) + \cancel{f_z(2-2)}$$

$$= 4(-0.1) + 1(+0.1)$$

$$= -0.3$$

$$\underline{f(2,1,2) = 1 \cdot e^0 = 1}$$

So $\boxed{f(1.9, 1.1, 2) \approx 0.7}$

(d) (10 points) Find the directional derivative of f at $(2,1,2)$ in the direction of $-\hat{\imath} + \hat{\jmath}$.

$$\vec{v} = -\hat{\imath} + \hat{\jmath} = \langle -1, 1, 0 \rangle \qquad |\vec{v}| = \sqrt{2}$$

$$\hat{u} = \frac{\vec{v}}{|\vec{v}|} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle$$

$$D_{\hat{u}} f(2,1,2) = \langle 4, 1, -4 \rangle \cdot \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle$$

$$= -\frac{4}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 = \boxed{-\frac{3}{\sqrt{2}}}$$

3. (25 points) Use Lagrange multipliers to find the closest point(s) to the origin along the curve $y^2 - 16x^2 = 9$. *Hint: instead of minimizing the distance to the origin, you might want to consider minimizing the **square** of the distance.*

$$f(x,y) = x^2 + y^2 \qquad \text{square of distance from } (x,y) \text{ to origin}$$

Want min $f(x,y)$ subject to $y^2 - 16x^2 = 9$

Step 1 $\nabla f = \langle 2x, 2y \rangle \qquad \nabla g = \langle -32x, 2y \rangle$
gradients

Step 2 Equations: $\nabla f = \lambda \nabla g \ \& \ g = c$

$$2x = -32\lambda x$$
$$2y = 2\lambda y$$
$$y^2 - 16x^2 = 9$$

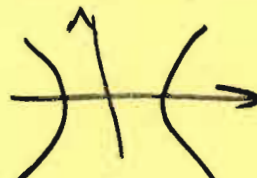**Step 3** solve: $2y = 2\lambda y \Rightarrow y = 0$ or $\lambda = 1$

$y = 0 \Rightarrow -16x^2 = 9$ (not possible)

$\lambda = 1 \Rightarrow x = 0 \Rightarrow y^2 = 9$ so $y = \pm 3 \Rightarrow (0,3) \ \& \ (0,-3)$

Step 4 values: $f(0,3) = 9 = f(0,-3)$

$y^2 - 16x^2 = 9$ hyperbola

so the distance has a min, on $g = c$

Thus $\boxed{(0,3) \ \& \ (0,-3)}$ are the closest points

4. Consider the function $f(x, y) = (x^2 - y^2)e^y$.

   (a) (10 points) In which direction should one go from the point $(-1, 0)$ to obtain the most rapid increase in f? Express your answer as a unit vector.

   direction of gradient $\nabla f (-1, 0)$

   $\nabla f = \langle 2x e^y, (x^2 - y^2) e^y - 2y e^y \rangle$

   $\nabla f (-1, 0) = \langle -2, 1 \rangle$ norm $= \sqrt{4+1} = \sqrt{5}$

   unit vector $\dfrac{\langle -2, 1 \rangle}{\sqrt{5}} = \boxed{\langle -\dfrac{2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}} \rangle}$

   (b) (20 points) Find all critical points of the function f and decide whether they are local maxima, local minima, or saddle.

   critical points: $\nabla f = 0$

   $2x e^y = 0$ $\Rightarrow x = 0$ $\Bigg\} \Rightarrow y^2 = -2y$

   $(x^2 - y^2 - 2y) e^y = 0 \Rightarrow x^2 - y^2 - 2y = 0$ $\qquad y = 0$ or

   $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad y = -2$

   critical points $\boxed{(0, 0) \; \& \; (0, -2)}$

   $H(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2e^y & 2x e^y \\ 2x e^y & (x^2 - y^2 - 2y)e^y + (-2y-2)e^y \end{bmatrix}$

   $H(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ det $H = -4 < 0$ so $\boxed{(0,0) \text{ saddle point}}$

   $H(0, -2) = \begin{bmatrix} 2 e^{-2} & 0 \\ 0 & 2e^{-2} \end{bmatrix}$ det $H = 4 e^{-4} > 0$ $\Bigg| \Rightarrow \boxed{(0, -2) \text{ local min}}$
   $\qquad\qquad\qquad\qquad\qquad\qquad\qquad f_{xx} = 2 e^{-2} > 0$