MATH 20C
Exam 2

Lecture C
November 12, 2010

Name: SOLUTIONS - pink version

PID: ______

TA: ______

Section #: ______ Section time: ______

There are 7 pages and 4 questions, for a total of 100 points.

No notes, no calculators, no books.

Please turn off all electronic devices.
Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. Unless otherwise stated, show all your work for full credit. To maximize credit, cross out incorrect work.

Good luck! ☺

Question:	1	2	3	4	Total
Points:	15	30	25	30	100
Score:					

1. (15 points) The variables x, y, z satisfy the relation $x^2yz^3 + z^4 = 24$. Find $\dfrac{\partial z}{\partial y}$ at the point $(1, 1, 2)$.

$$x^2 y z^3 + z^4 = 24$$

Take $\dfrac{\partial z}{\partial y}$ of both sides of the equation

$$x^2 \cdot \left[1 \cdot z^3 + y\left(3z^2 \frac{\partial z}{\partial y}\right)\right] + 4z^3 \frac{\partial z}{\partial y} = 0$$

Substitute $x=1, y=1, z=2$

$$1\left[8 + 12\frac{\partial z}{\partial y}\right] + 32\frac{\partial z}{\partial y} = 0$$

$$8 + 44\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{8}{44} = \boxed{-\frac{2}{11}}$$

2. Let $f(x, y, z) = y\sin(x - z) + xy^3$.

(a) (5 points) Find ∇f at $(2, 1, 2)$.

$$\nabla f = \langle f_x, f_y, f_z \rangle =$$
$$= \langle y\cos(x-z) + y^3, \sin(x-z) + 3xy^2, -y\cos(x-z) \rangle$$

$$\nabla f(2,1,2) = \langle 1\cos 0 + 1, \sin 0 + 6, -1\cos(0) \rangle$$
$$= \langle 2, 6, -1 \rangle$$

(b) (5 points) Write the equation for the tangent plane to the surface $f = 2$ at the point $(2, 1, 2)$.

normal vector: $\nabla f(2,1,2) = \langle 2, 6, -1 \rangle$

passes through $(2,1,2)$

$$2x + 6y - z = 2\cdot 2 + 6\cdot 1 - 2$$
$$\boxed{2x + 6y - z = 8}$$

(c) (10 points) Use a linear approximation to find the approximate value of $f(1.9, 1.1, 2)$.

$$\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z$$

$$f(1.9, 1.1, 2) - f(2,1,2) \approx$$

$$\approx 2\Delta x + 6\Delta y - \Delta z$$

$$= 2(1.9-2) + 6(1.1-1) - \cancel{(2-2)}$$

$$= 2(-0.1) + 6(0.1)$$

$$= 0.4$$

$$f(2,1,2) = 1 \sin 0 + 2 \cdot 1 = 2$$

So $\boxed{f(1.9, 1.1, 2) \approx 2.4}$

(d) (10 points) Find the directional derivative of f at $(2, 1, 2)$ in the direction of $-\hat{\imath} + \hat{\jmath}$.

$\vec{v} = \langle -1, 1, 0 \rangle$ $|\vec{v}| = \sqrt{2}$

unit vector $\hat{u} = \dfrac{\vec{v}}{|\vec{v}|} = \left\langle -\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, 0 \right\rangle$

$$D_{\hat{u}} f = \nabla f(2,1,2) \cdot \hat{u} = \langle 2, 6, -1 \rangle \cdot \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$$

$$= -\frac{2}{\sqrt{2}} + \frac{6}{\sqrt{2}} + 0 = \frac{4}{\sqrt{2}} = \boxed{2\sqrt{2}}$$

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3. (25 points) Use Lagrange multipliers to find the closest point(s) to the origin along the curve $4x^2 + 9y^2 = 16$. *Hint: instead of minimizing the distance to the origin, you might want to consider minimizing the* **square** *of the distance.*

$f(x,y) = x^2+y^2$ is the square of the distance from (x,y) to the origin

Want min $f(x,y)$ subject to $4x^2+9y^2=16$

$g(x,y) = 4x^2+9y^2$

Step 1 ~~∇f~~ Gradients

$\nabla f = \langle 2x, 2y \rangle \qquad \nabla g = \langle 8x, 18y \rangle$

Step 2 Equations $\nabla f = \lambda \nabla g$, $g = c$

$2x = 8\lambda x \Rightarrow x = 0$ or $\lambda = \frac{1}{4}$

$2y = 18\lambda y$

Step 3 solve $4x^2+9y^2=16$

$x=0 \Rightarrow 9y^2 = 16 \Rightarrow y = \pm\frac{4}{3}$ so $(0,\frac{4}{3})$ $(0,-\frac{4}{3})$

$\lambda = \frac{1}{4} \Rightarrow 2y = \frac{9}{2}y \Rightarrow y = 0 \Rightarrow 4x^2 = 16 \Rightarrow x = \pm 2$

So $(2,0)$ $(-2,0)$

Step 4 values $f(2,0) = f(-2,0) = 4$

$f(0,\frac{4}{3}) = f(0,-\frac{4}{3}) = \frac{16}{9} < 4$

$4x^2+9y^2=16$ is closed, so minimum is at one of these points. Since $\frac{16}{9} < 4$, $\boxed{(0, \frac{4}{3}) \ \& \ (0,-\frac{4}{3})}$ are the points closest to the origin.

4. Consider the function $f(x, y) = x^3 - 3xy + y^3$.

(a) (10 points) In which direction should one go from the point $(1, 0)$ to obtain the most rapid increase in f? Express your answer as a unit vector.

In the direction of the gradient $\nabla f(1,0)$

$\nabla f = \langle 3x^2 - 3y, -3x + 3y^2 \rangle$

$\nabla f(1,0) = \langle 3, -3 \rangle$ $|\nabla f(1,0)| = \sqrt{9+9} = 3\sqrt{2}$

$\hat{u} = \left\langle \frac{3}{3\sqrt{2}}, -\frac{3}{3\sqrt{2}} \right\rangle = \boxed{\left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle}$

(b) (20 points) Find all critical points of the function f and decide whether they are local maxima, local minima, or saddle.

critical points : $\nabla f = 0$: $3x^2 - 3y = 0 \Rightarrow x^2 = y$

$-3x + 3y^2 = 0 \Rightarrow x = y^2$

So $x = y^2 = (x^2)^2 \Rightarrow x = x^4 \Rightarrow x(x^3 - 1) = 0$

$\underline{x = 0} \Rightarrow y = 0$

$\underline{x = 1} \Rightarrow y = 1^2 = 1$

$\boxed{(0,0) \quad (1,1)}$ critical points

$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$

$H(0,0) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$ $\det H = -9 < 0 \Rightarrow$ $\boxed{(0,0) \text{ saddle point}}$

$H(1,1) = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$ $\det H = 36 - 9 = 27 > 0$, $f_{xx} = 6 > 0$ $\Rightarrow$ $\boxed{(1,1) \text{ local min}}$