MATH 20C | Lecture C
Exam 2 | November 12, 2010

Name: SOLUTIONS - green version

PID: ____________

TA: ____________

Section #: ________ Section time: ____________

There are 7 pages and 4 questions, for a total of 100 points.

No notes, no calculators, no books.

Please turn off all electronic devices.
Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. Unless otherwise stated, show all your work for full credit. To maximize credit, cross out incorrect work.

Good luck! ☺

Question:	1	2	3	4	Total
Points:	15	30	25	30	100
Score:					

Name: ______________________

Section #: Section time:

1. (15 points) The variables x, y, z satisfy the relation $xyz + z^3 = 10$. Find $\dfrac{\partial z}{\partial y}$ at the point $(1, 1, 2)$.

$$xyz + z^3 = 10$$

Take $\dfrac{\partial}{\partial y}$ of both sides

$$x\left[1 \cdot z + y\frac{\partial z}{\partial y}\right] + 3z^2\frac{\partial z}{\partial y} = 0$$

Substitute $x = 1$, $y = 1$, $z = 2$

$$1\left[2 + \frac{\partial z}{\partial y}\right] + 12\frac{\partial z}{\partial y} = 0$$

$$2 + 13\frac{\partial z}{\partial y} = 0$$

$$\boxed{\frac{\partial z}{\partial y} = -\frac{2}{13}}$$

2. Let $f(x, y, z) = x^2 + 2xyz^3 + 2y^2$.

(a) (5 points) Find ∇f at $(2, 1, 2)$.

$\nabla f = \langle 2x + 2yz^3,\ 2xz^3 + 4y,\ 6xyz^2 \rangle$

$\nabla f(2,1,2) = \langle 4 + 2\cdot 8,\ 32 + 4,\ 48 \rangle$

$= \boxed{\langle 20, 36, 48 \rangle}$

(b) (5 points) Write the equation for the tangent plane to the surface $f = 38$ at the point $(2, 1, 2)$.

Gradient = normal vector

Sol I

$20x + 36y + 48z = 20\cdot 2 + 36\cdot 1 + 48\cdot 2$

$= 40 + 36 + 96$

$\boxed{20x + 36y + 48z = 172}$

Sol II Scale down by 4 $\nabla f = \langle 5, 9, 12 \rangle \cdot 4$

normal vector : $\langle 5, 9, 12 \rangle$

=

$\boxed{5x + 9y + 12z = 43}$

(c) (10 points) Use a linear approximation to find the approximate value of $f(1.9, 1.1, 2)$.

$$f(1.9, 1.1, 2) - f(2,1,2) \approx f_x \Delta x + f_y \Delta y + f_z \Delta z$$
$$= f_x\,(1.9-2) + f_y\,(1.1-1) + \cancel{f_z\,(2-2)}$$
$$= 20(-0.1) + 36(0.1)$$
$$= 3.6 - 2 = 1.6$$

$f(2,1,2) = 38$ (see point b)

So $\boxed{f(1.9, 1.1, 2) \approx 39.6}$

(d) (10 points) Find the directional derivative of f at $(2,1,2)$ in the direction of $-\hat{\imath}+\hat{\jmath}$.

$\vec{v} = \langle -1, 1, 0\rangle \qquad |\vec{v}| = \sqrt{2}$

$\hat{u} = \dfrac{\vec{v}}{|\vec{v}|} = \left\langle -\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, 0\right\rangle$

$D_{\hat{u}} f = \langle 20, 36, 48\rangle \cdot \left\langle -\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, 0\right\rangle$

$= -\dfrac{20}{\sqrt{2}} + \dfrac{36}{\sqrt{2}} = \dfrac{16}{\sqrt{2}} = \boxed{8\sqrt{2}}$

3. (25 points) Use Lagrange multipliers to find the closest point(s) to the origin along the curve $4x^2 - y^2 = 9$. *Hint: instead of minimizing the distance to the origin, you might want to consider minimizing the* **square** *of the distance.*

$f(x,y) = x^2+y^2$ square of distance from (x,y) to origin

Want Minimize $f(x,y)$ subject to $4x^2-y^2=9$

$g(x,y) = 4x^2-y^2$

Step 1 Gradients : $\nabla f = \langle 2x, 2y \rangle$ $\nabla g = \langle 8x, -2y \rangle$

Step 2 Equations $\nabla f = \lambda \nabla g$ & $g = c$

$2x = 8\lambda x$

$2y = -2\lambda y$

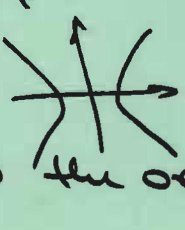
$4x^2-y^2=9$

Step 3 Solve : $2y = -2\lambda y \Rightarrow$ $y=0 \Rightarrow 4x^2=9 \Rightarrow x = \pm\frac{3}{2}$; $\lambda=-1 \Rightarrow x=0 \Rightarrow -y^2=9$ (not possible)

So $\left(\frac{3}{2},0\right)$ & $\left(-\frac{3}{2},0\right)$ are the points I'm looking for

Step 4 Values: $f\left(\frac{3}{2},0\right) = \frac{9}{4} = f\left(-\frac{3}{2},0\right)$

$4x^2-y^2=9$ is a hyperbola



so there are points closest to the origin $\Rightarrow f$ attains its minimum

Hence $\boxed{\left(\frac{3}{2},0\right), \left(-\frac{3}{2},0\right)}$ are the closest pts to origin

4. Consider the function $f(x, y) = (y^2 - x^2)e^x$.

   (a) (10 points) In which direction should one go from the point $(0, 1)$ to obtain the most rapid increase in f? Express your answer as a unit vector.

direction of $\nabla f(0,1)$

$\nabla f = \langle -2xe^x + (y^2-x^2)e^x, 2ye^x \rangle$

$\nabla f(0,1) = \langle 1, 2 \rangle$ norm $= \sqrt{1+4} = \sqrt{5}$

unit vector $\dfrac{\langle 1,2\rangle}{\sqrt{5}} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

   (b) (20 points) Find all critical points of the function f and decide whether they are local maxima, local minima, or saddle.

critical points : $\nabla f = 0$

$2ye^x = 0 \Rightarrow y = 0$

$(y^2 - x^2 - 2x)e^x = 0$

$\Rightarrow -x^2 - 2x = 0 \Rightarrow x = 0$ or $x = -2$

So $\boxed{(0,0) \text{ and } (-2,0)}$ are the critical points

$$H(x,y) = \begin{bmatrix} (y^2-x^2-2x)e^x + (-2x-2)e^x & 2ye^x \\ 2ye^x & 2e^x \end{bmatrix}$$

$H(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ $\det H = -4 < 0 \Rightarrow \boxed{(0,0) \text{ saddle point}}$

$H(-2,0) = \begin{bmatrix} +2e^{-2} & 0 \\ 0 & 2e^{-2} \end{bmatrix}$ $\det H = 4e^{-4} > 0$

$f_{xx} = 2e^{-2} > 0$

so $\boxed{(-2,0) \text{ local min}}$