MATH 20C
Exam 2

Lecture C
November 12, 2010

Name: SOLUTIONS- blue version

PID: ____________________

TA: ____________________

Section #: ________ Section time: ____________________

There are 7 pages and 4 questions, for a total of 100 points.

No notes, no calculators, no books.

Please turn off all electronic devices.
Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. Unless otherwise stated, show all your work for full credit. To maximize credit, cross out incorrect work.

Good luck! ☺

Question:	1	2	3	4	Total
Points:	15	30	25	30	100
Score:					

1. (15 points) The variables x, y, z satisfy the relation $x^2y^2z^2 + z^3 = 12$. Find $\dfrac{\partial z}{\partial y}$ at the point $(1, 1, 2)$.

$$x^2 y^2 z^2 + z^3 = 12$$

Take $\dfrac{\partial}{\partial y}$ of both sides.

Get:

$$x^2\left[(2y)z^2 + y^2\left(2z\frac{\partial z}{\partial y}\right)\right] + 3z^2\frac{\partial z}{\partial y} = 0$$

Substitute $x=1, y=1, z=2$

$$1^2\left[(2\cdot 1)\,4 + 1^2\cdot\left(2\cdot 2\cdot\frac{\partial z}{\partial y}\right)\right] + 12\frac{\partial z}{\partial y} = 0$$

$$8 + 16\frac{\partial z}{\partial y} = 0$$

$$\boxed{\frac{\partial z}{\partial y} = -\frac{1}{2}}$$

2. Let $f(x, y, z) = x^2y^2 - x + yz^2$.

   (a) (5 points) Find ∇f at $(2, 1, 2)$.

$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2xy^2 - 1, 2x^2y + z^2, 2yz \rangle$

at $(2,1,2)$

$\nabla f(2,1,2) = \langle 4-1, 8+4, 4 \rangle = \boxed{\langle 3, 12, 4 \rangle}$

   (b) (5 points) Write the equation for the tangent plane to the surface $f = 6$ at the point $(2, 1, 2)$.

$\nabla f(2,1,2) = \langle 3, 12, 4 \rangle$ normal vector

plane passes through $(2,1,2)$ so

$3x + 12y + 4z = 3 \cdot 2 + 12 \cdot 1 + 4 \cdot 2$

$\boxed{3x + 12y + 4z = 26}$

(c) (10 points) Use a linear approximation to find the approximate value of $f(1.9, 1.1, 2)$.

$$\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z$$

$$f(1.9, 1.1, 2) - f(2,1,2) \approx f_x\,(1.9-2) + f_y\,(1.1-1) + \cancel{f_z\,(2-2)}$$

$$= 3 \cdot (-0.1) + 12 \cdot 0.1 = 0.9$$

$$f(2,1,2) = 4 \cdot 1 - 2 + 1 \cdot 4 = 6$$

So $\boxed{f(1.9, 1.1, 2) \approx 6 + 0.9 = 6.9}$

(d) (10 points) Find the directional derivative of f at $(2, 1, 2)$ in the direction of $-\hat{\imath} + \hat{\jmath}$.

$$\vec{v} = -\hat{\imath} + \hat{\jmath} = \langle -1, 1, 0 \rangle \quad \to \quad |\vec{v}| = \sqrt{2}$$

$$\hat{u} = \frac{|\vec{v}|}{|\vec{v}|} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$$

$$D_{\hat{u}} f(P) = \nabla f(P) \cdot \hat{u} = \langle 3, 12, 4 \rangle \cdot \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$$

$$= -\frac{3}{\sqrt{2}} + \frac{12}{\sqrt{2}} + 0 = \boxed{\frac{9}{\sqrt{2}}}$$

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3. (25 points) Use Lagrange multipliers to find the closest point(s) to the origin along the curve $4x^2 + y^2 = 9$. *Hint: instead of minimizing the distance to the origin, you might want to consider minimizing the* **square** *of the distance.*

$f(x,y) = x^2 + y^2$ the square of the distance from (x,y) to the origin.

Want to minimize $f(x,y)$ under constraint

$$4x^2 + y^2 = 9$$

$$g(x,y) = 4x^2 + y^2$$

Step 1 Gradients

$$\nabla f = \langle 2x, 2y \rangle$$

$$\nabla g = \langle 8x, 2y \rangle$$

Step 2 Equations $\nabla f = \lambda \nabla g$ + $g = c$

$$2x = 8\lambda x$$

$$2y = 2\lambda y$$

$$4x^2 + y^2 = 9$$

Step 3 Solve $2y(1-\lambda) = 0 \Rightarrow y = 0$ or $\lambda = 1$

If $\lambda = 1 \Rightarrow 2x = 8x \Rightarrow x = 0$, $4x^2 + y^2 = 9$ $\Rightarrow y^2 = 9$ $(0,3), (0,-3)$

If $y = 0 \Rightarrow 4x^2 = 9 \Rightarrow x = \pm \frac{3}{2} \Rightarrow (\frac{3}{2}, 0), (-\frac{3}{2}, 0)$

Step 4 Values: $f(0,3) = 9 = f(0,-3)$

$$f(\tfrac{3}{2}, 0) = \frac{9}{4} = f(-\tfrac{3}{2}, 0)$$

$4x^2 + y^2 = 9$ closed $\Rightarrow$ minimum is one of these points.

Looking at values, see closest points are $\boxed{(\frac{3}{2}, 0), (-\frac{3}{2}, 0)}$

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4. Consider the function $f(x, y) = x^3 + xy + y^3$.

(a) (10 points) In which direction should one go from the point $(0, 1)$ to obtain the most rapid increase in f? Express your answer as a unit vector.

In the direction of the gradient at $(0,1)$

$\nabla f = \langle 3x^2 + y, x + 3y^2 \rangle$

$\nabla f(0,1) = \langle 1, 3 \rangle$

unit vector in this direction $\frac{\langle 1,3 \rangle}{\sqrt{1+9}} = \boxed{\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \rangle}$

(b) (20 points) Find all critical points of the function f and decide whether they are local maxima, local minima, or saddle.

$\nabla f = 0 \Rightarrow 3x^2 + y = 0 \Rightarrow y = -3x^2$

$x + 3y^2 = 0$

Sub 1st into 2nd $\Rightarrow x + 3(-3x^2)^2 = 0$

$x + 27x^4 = 0 \Rightarrow x = 0$ or $x^3 = -\frac{1}{27} \Rightarrow x = -\frac{1}{3}$

$x = 0 \Rightarrow y = 0$

$x = -\frac{1}{3} \Rightarrow y = -3 \cdot \frac{1}{9} = -\frac{1}{3}$

$\boxed{(0,0) \quad (-\frac{1}{3}, -\frac{1}{3})}$ critical points

$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & 1 \\ 1 & 6y \end{bmatrix}$

At $(0,0)$ $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \det H < 0 \Rightarrow$ $\boxed{(0,0) \text{ saddle point}}$

At $(-\frac{1}{3}, -\frac{1}{3})$ $H = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \Rightarrow \left. \begin{array}{l} \det H = 3 > 0 \\ f_{xx} = -2 < 0 \end{array} \right\} \Rightarrow$ $\boxed{(-\frac{1}{3}, -\frac{1}{3}) \text{ local max}}$