Section 15.3 # 26

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We are asked to evaluate $\iiint_{\mathcal{W}} y\ dV$, where $\mathcal{W}$ is the domain bounded by the elliptic cylinder $\frac{x^2}{4} + \frac{y^2}{9} = 1$, and the sphere $x^2 + y^2 + z^2 = 16$, all in the first octant. So this means $x, y, z \geq 0$.

So here is my solution. It might not be the easiest. Here is a list of other things to try:

- Try changing the order of integration. Maybe integrate with respect to y first? Be careful, though! You need to figure out the the bounds again.
- After integrating with respect to z, try polar coordinates for the xy plane. Careful! Here $dA = rdrd\theta$.! Hint: the bounds of integration are $0 \leq \theta \leq \pi/2$, and $0 \leq r \leq (\cos^2\theta/4 + \sin^2\theta/9)^{-1/2}$. Think about why this is true.

So here it goes. Warning: Work out the details for yourself! For two reasons: (1) I may be wrong (it has been known to happen). (2) You won't learn anything if you just write down my solution...

Lets describe the region in the xy plane first. We choose the outer limits of integration to represent x going from 0 to 2. Now pick some x between 0 and 2. Given that x, we want to figure out what y runs between. The lowe limit for y is the easy part: it has to be zero. What about the upper limit? Well if we solve $\frac{x^2}{4} + \frac{y^2}{9} = 1$ for y, then we will get the bound on y given some x. Solving for y gives $y = \pm 3\sqrt{1 - \frac{x^2}{4}}$. But we know that we want to be in the first octant, so $y \geq 0$. So we want the positive radical. So given an x, y goes from 0 to $3\sqrt{1 - \frac{x^2}{4}}$.

So this is what is going on in the xy plane. We now need to extend in the z axis. So ask: given some x and y in the region we just defined above, what does z go between? Again, since we are in the first octant, the lower limit of z is 0. And like before the upper limit of z comes from solving $x^2 + y^2 + z^2 = 1$ in terms of z. This is $z = \pm\sqrt{16 - x^2 - y^2}$. Again, since we are in the first octant, we want the positive root. So z runs from 0 to $\sqrt{16 - x^2 - y^2}$.

Putting this all together:

$$\iiint_{\mathcal{W}} y\ dV = \int_0^2 \int_0^{3\sqrt{1-\frac{x^2}{4}}} \int_0^{\sqrt{16-x^2-y^2}} y\ dzdydx \tag{1}$$

So we start with the inner most integral, which says to integrate y with respect to z. But y is a constant with respect to z, so its antiderivative is yz. So then (1) is equal to:

$$\int_0^2 \int_0^{3\sqrt{1-\frac{x^2}{4}}} y\sqrt{16-x^2-y^2}\ dydx \tag{2}$$

Lets think about the inner integral.

$$\int y\sqrt{16-x^2-y^2}dy \tag{3}$$

We can do this by u substitution. Let $u = 16-x^2-y^2$. Then $du = -2ydy$. Remember that we are treating x like a constant. So then

$$\int y\sqrt{16-x^2-y^2}\ dy = -\frac{1}{3}(16-x^2-y^2)^{3/2} \tag{4}$$

This is just the antiderivative. We need to evaluate it at $y = 3\sqrt{1-\frac{x^2}{4}}$, and $y = 0$ and take their difference. So we end up with (1) equals

$$-\frac{1}{3}\int_0^2 \left[\left(16-x^2-\left(3\sqrt{1-\frac{x^2}{4}}\right)^2\right)^{3/2} - (16-x^2)^{3/2}\right]\ dx \tag{5}$$

Which simplifies to:

$$-\frac{1}{3}\int_0^2 \left[\left(7-\frac{13x^2}{4}\right)^{3/2} - (16-x^2)^{3/2}\right]\ dx \tag{6}$$

And this can be rewritten

$$\frac{64}{3}\int_0^2 \left(1-\frac{1}{16}x^2\right)^{3/2}\ dx - \frac{7^{3/2}}{3}\int_0^2 \left(1-\frac{13}{28}x^2\right)^{3/2}\ dx \tag{7}$$

At this point, the only thing left to do is figure out what the antiderivative of $\int(1-ax^2)^{3/2}\ dx$ is. Lets make a substitution. Say that $x = 1/\sqrt{a}\sin\theta$. Then $dx = 1/\sqrt{a}\cos\theta$. So

$$\int(1-ax^2)^{3/2}\ dx = 1/\sqrt{a}\int\cos^4\theta\ d\theta \tag{8}$$

And we have tables to tell us what the integral of $\cos^4\theta$ is. Apparently:

$$\int \cos^4\theta \; d\theta = \frac{1}{32}(12\theta + 8\sin(2\theta) + \sin(4\theta)) \tag{9}$$

So now we are basically done. All that remains is to evaluate (7) using the antiderivative we just discovered. Remember to be careful to change the bounds of integration, since the above equation is in terms of θ, but we need it to be in terms of x. Recall that the substitution we used was $x = 1/\sqrt{a}\sin\theta$.