MATH 20C – MIDTERM 2
SOLUTIONS TO PRACTICE PROBLEMS

ALINA BUCUR

Problem 1: 4 (use change of variables $x^2 + y^2 + 1 = t$).

Problem 2: (a) I only meant to ask for the traces through the origin, Namely the horizontal trace for $z = 0$ (which consists of the y-axis and the graph of the function $y = x^3$ in the xy-plane), the vertical traces when $x = 0$ (the y-axis in the yz-plane) and when $y = 0$ (the graph of the function $z = -x^4$ in the xz-plane; this is a steeper upside-down parabola).

(b) $\nabla f = \langle y - 4x^3, x\rangle$, so at $(1, 1)$ it becomes $\langle -3, 1\rangle$.

(c) $\Delta w \approx -3\Delta x + \Delta y$.

Problem 3: (a) $f(x, y, z) = x^3y + z^2$, so $\nabla f = \langle 3x^2y, x^3, 2z\rangle = \langle 3, -1, 4\rangle$. The tangent plane is $3x - y + 4z = 4$.

(b) $f(x, y, z) = 3x^2 - 4y^2 - z$, so $\nabla f = \langle 6x, -8y, -1\rangle$. We want this to be a scalar multiple of $\langle 3, 2, 2\rangle$, hence we want $6x = 3\lambda, -8y = 2\lambda, -1 = 2\lambda$. The point is $(-1/4, 1/8, 1/8)$.

Problem 4: (a) The volume is $xyz = xy(1 - x^2 - y^2) = xy - x^3y - xy^3$. Critical points: $f_x = y - 3x^2y - y^3 = 0, f_y = x - x^3 - 3xy^2 = 0$.

(b) Assuming $x > 0$ and $y > 0$, the equations can be rewritten as $1 - 3x^2 - y^2 = 0, 1 - x^2 - 3y^2 = 0$. Solution: $x^2 = y^2 = 1/4$, i.e. $(x, y) = (1/2, 1/2)$.

(c) $H(x, y) = \begin{bmatrix} -6xy & 1 - 3x^2 - 3y^2 \\ 1 - 3x^2 - 3y^2 & -6xy \end{bmatrix}$, so $H(1/2, 1/2) = \begin{bmatrix} -3/2 & -1/2 \\ -1/2 & -3/2 \end{bmatrix}$. Since $\det H > 0$ and $f_{xx} < 0$, the point is a local maximum.

(d) The maximum of f lies either at $(1/2, 1/2)$, or on the boundary of the domain or at infinity. Since $f(x, y) = xy(1 - x^2 - y^2)$, $f = 0$ when either $x \to 0$ or $y \to 0$, and $f \to -\infty$ when $x \to \infty$ or $y \to \infty$ (since $x^2 + y^2 \to \infty$) . So the maximum is at $(x, y) = (1/2, 1/2)$, where $f(1/2, 1/2) = 1/8$.

Problem 5: (a) $f(x, y, z) = xyz$ and $g(x, y, z) = x^2 + y^2 + z = 1$: one must solve the Lagrange multiplier equation $\nabla f = \lambda \nabla g$, i.e. $yz = 2\lambda x, xz = 2\lambda y, xy = \lambda$, and the constraint equation $x^2 + y^2 + z = 1$.

(b) Dividing the first two equations by each other get $y/x = x/y$, so $x^2 = y^2$; since $x > 0, y > 0$ we get $x = y$. Substituting this into the Lagrange multiplier equations, we get $z = 2\lambda$ and $x^2 = \lambda$. Hence $z = 2x^2$ and the constraint equation becomes $4x^2 = 1$. Thus $x = 1/2, y = 1/2, z = 1/2$.

Problem 6: $\dfrac{\partial w}{\partial x} = f_u u_x + f_v v_x = y f_u + \dfrac{1}{y} f_v$ and $\dfrac{\partial w}{\partial y} = f_u u_y + f_v v_y = x f_u - \dfrac{x}{y^2} f_v$.

Problem 7: Let $f(x, y) = x^2y^2 - x$.

(a) $\nabla f = \langle 2xy^2 - 1, 2x^2y\rangle$, so at $(2,1)$ it becomes $\langle 3, 8\rangle$.

(b) $z - 2 = 3(x-2) + 8(y-1)$ or $3x + 8y - z = 12$.

(c) $\Delta x = 1.9 - 2 = -0.1$ and $\Delta y = 1.1 - 1 = 0.1$; so $f(1.9, 1.1) - f(2,1) \approx 3\Delta x + 8\Delta y = -0.3 + 0.8 = 0.5$; since $f(2,1) = 2$, we obtain $f(1.9, 1.1) \approx 2.5$

(d) $\hat{u} = \dfrac{\langle -1, 1\rangle}{|\langle -1, 1\rangle|} = \left\langle -\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}} \right\rangle$. The directional derivative is

$$D_{\hat{u}} f = \nabla f \cdot \hat{u} = \langle 3, 8\rangle \cdot \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{5}{\sqrt{2}}.$$

Problem 8: (a) $w_x = -6x - 4y + 16 = 0 \implies -3x - 2y + 8 = 0$ and $w_y = -4x - 2y - 12 = 0 \implies 4x + 2y + 12 = 0$. So $x = -20, y = 34$. Therefore there is just one critical point at $(-20, 34)$. Now $H(x,y) = \begin{bmatrix} -6 & -4 \\ -4 & -2 \end{bmatrix}$, so $\det H < 0$ and the critical point is a saddle point.

(b) There is no critical point in the first quadrant, hence the maximum must be at infinity or on the boundary of the first quadrant.
The boundary is composed of two half-lines:
- $x = 0$ and $y \geq 0$, on which $w = -y^2 - 12y$. It has a maximum ($w = 0$) at $y = 0$.
- $y = 0$ and $x \geq 0$, on which $w = -3x^2 + 16x$ (downwards parabola). It has a maximum when $w_x = -6x + 16 = 0 \implies x = 8/3$. Hence w has a local maximum at $(8/3, 0)$ and the value is $w = -3(8/3)^2 + 16(8/3) = 64/3 > 0$.

We now check that the maximum of w is not at infinity.
- $y \geq 0$ and $x \to \infty : w \leq -3x^2 + 16x$, which tends to $-\infty$ as $x \to +\infty$.
- $0 \leq x \leq C$ and $y \to \infty : w \leq -y^2 + 16C$, which tends to $-\infty$ as $y \to +\infty$.

We conclude that the maximum of w in the first quadrant is at $(8/3, 0)$.

Problem 9: Let $u = y/x$, $v = x^2 + y^2$, $w = w(u,v)$.

(a) $w_x = w_u u_x + w_v v_x = -\dfrac{y}{x^2} w_u + 2x w_v$ and $w_y = w_u u_y + w_v v_y = \dfrac{1}{x} w_u + 2y w_v$.

(b) $x w_x + y w_y = x\left(-\dfrac{y}{x^2} w_u + 2x w_v\right) + y\left(\dfrac{1}{x} w_u + 2y w_v\right) = \left(-\dfrac{y}{x} + \dfrac{y}{x}\right) w_u + (2x^2 + 2y^2) w_v = 2v w_v$

(c) $x w_x + y w_y = 2v w_v = 2v(5v^4) = 10v^5$.

Problem 10: (a) $f(x,y,z) = x$ with constraint $g(x,y,z) = x^4 + y^4 + z^4 + xy + yz + zx = 6$. The Lagrange multiplier equation is

$$\nabla f = \lambda \nabla g \Leftrightarrow \begin{cases} 1 = \lambda(4x^3 + y + z) \\ 0 = \lambda(4y^3 + x + z) \\ 0 = \lambda(4z^3 + x + y). \end{cases}$$

(b) The level surfaces of f and g are tangent at (x_0, y_0, z_0), so they have the same tangent plane. The level surface of f is the plane $x = x_0$; hence this is also the tangent plane to the surface $g = 6$ at (x_0, y_0, z_0).

Second method: at $((x_0, y_0, z_0)$, we have $1 = \lambda g_x, 0 = \lambda g_y, 0 = \lambda g_z$. So $\lambda \neq 0$ and $\langle g_x, g_y, g_z\rangle = \langle 1/\lambda, 0, 0\rangle$. This vector is therefore perpendicular to the tangent plane to the surface at (x_0, y_0, z_0). The equation of the plane is then $\frac{1}{\lambda}(x - x_0) = 0$, or equivalently $x = x_0$.

Problem 11: The tangent line is the intersection of the tangent planes to the two surfaces. Therefore it is parallel to the cross product of their normal vectors, $\nabla f \times \nabla g$. Since $\nabla f = \langle 2x, 3y^2, -4z^3\rangle = \langle 2, 3, -4\rangle$ and $\nabla g = \langle y + z, x, 3z^2 + x\rangle = \langle 2, 1, 4\rangle$, the direction of the tangent line is given by $\langle 2, 3, -4\rangle \times \langle 2, 1, 4\rangle = \langle 16, -16, -4\rangle$. Therefore the equation is

$$\vec{L}(s) = \langle 1, 1, 1\rangle + s\langle 16, -16, -4\rangle.$$

Problem 12: $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{1}{2xy} - \frac{1}{xy(xy+2)}\right) = \lim_{(x,y)\to(0,0)} \frac{xy}{2xy(xy+2)} = \lim_{(x,y)\to(0,0)} \frac{1}{2(xy+2)} = \frac{1}{4}.$