MATH 20C – ANSWERS TO THE NEW PROBLEMS ON THE STUDY GUIDE

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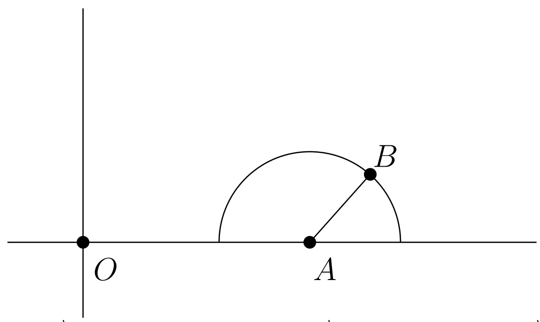
For the other practice problems, see the study guides for the two midterms.

Problem 11:



(a) $\overrightarrow{AB} = \langle \cos t, \sin t\rangle$ and $\overrightarrow{OA} = \langle 10t, 0\rangle$, so $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \langle 10t + \cos t, \sin t\rangle$.

The rear bumper is reached at time $t = \pi$ and the position of B is $(10\pi - 1, 0)$.

(b) $\vec{v}(t) = \langle 10 - \sin t, \cos t\rangle$, so

$$|\vec{v}|^2 = (10 - \sin t)^2 + \cos^2 t = 100 - 20 \sin t + \sin^2 t + \cos^2 t = 101 - 20 \sin t.$$

The speed is then given by $|\vec{v}| = \sqrt{101 - 20 \sin t}$.
The speed is smallest when $\sin t$ is largest i.e. $\sin t = 1$. It occurs when $t = \pi/2$. At this time, the position of the bug is $(5\pi, 1)$.
The speed is largest when $\sin t$ is smallest; that happens at the times $t = 0$ or π for which the position is then $(0, 0)$ and $(10\pi - 1, 0)$.

Problem 28:

$$\text{Mass}(R) = \iiint_R \rho(x, y, z) dV = \iiint_R y dV.$$

The equation of the sphere is $x^2 + y^2 + (z - 2)^2 = 16$.

The shadow of R on the xy-plane is given by the quarter of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ that sits in the first quadrant $x, y \geq 0$. So $0 \leq x \leq 2$ and for each x we have $0 \leq y \leq 3\sqrt{1 - \frac{x^2}{4}}$. For each point (x, y) in the shadow of R, have $0 \leq z \leq 2 + \sqrt{16 - x^2 - y^2}$. Hence

$$\text{Mass}(R) = \int_0^2 \int_0^{3\sqrt{1-\frac{x^2}{4}}} \int_0^{2+\sqrt{16-x^2-y^2}} y \, dz \, dy \, dx.$$

Problem 29:

The two surfaces are paraboloids (similar to Example 1 in Lecture 24 on Nov 19). The shadow of the region on the xy-plane is determined by the intersection of these two paraboloids. In other words, we need $z = 4 - x^2 - y^2$ to sit underneath $z = 10 - 4x^2 - 4y^2$, i.e. $4 - x^2 - y^2 \leq 10 - 4x^2 - 4y^2$. That is, we need $3x^2 + 3y^2 \leq 6 \Leftrightarrow x^2 + y^2 \leq 2$. So,

$$\text{vol} = \iint_{x^2+y^2\leq 2} \int_{4-x^2-y^2}^{10-4x^2-4y^2} dV$$

From here it's best to switch to cylindrical coordinates, so

$$\text{vol} = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{4-r^2}^{10-4r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} r(6-3r^2)\,dr\,d\theta = \int_0^{2\pi}\left[3r^2 - \frac{3}{4}r^4\right]_{r=0}^{r=\sqrt{2}} d\theta = 6\pi.$$

Problem 30: The region R is the triangle formed by the lines $y = x\sqrt{3}$, $y = x$ and $x = 2$.

The angle made by the line $y = x\sqrt{3}$ with the positive x-axis is $\pi/3$, while the angle made by the line $y = x$ with the positive x-axis is $\pi/4$. The line $x = 2$ crosses the two lines at $(2, 2\sqrt{3})$ and $(2, 2)$. The line $x = 2$ is given in polar coordinates by $r\cos\theta = 2$, hence $r = \frac{2}{\cos\theta}$.

$$\int_0^2\int_x^{x\sqrt{3}} x\,dy\,dx = \int_{\pi/4}^{\pi/3}\int_0^{2/\cos\theta} = r^2\cos\theta\,dr\,d\theta = \frac{8}{3}\int_{\pi/4}^{\pi/3}\frac{1}{\cos^2\theta} = \frac{8}{3}\Big[\tan\theta\Big]_{\theta=\pi/4}^{\theta=\pi/3} = \frac{8}{3}(\sqrt{3}-1).$$

Problem 31:

(a) The region of integration is the triangle made by the lines $y = x$, $y = 2x$ and $x = 1$. It has vertices $(0, 0)$, $(1, 1)$ and $(1, 2)$.

(b) For $0 \le y \le 1$, have $y/2 \le x \le y$ and for $1 \le y \le 2$, have $y/2 \le x \le 1$. So

$$\int_0^1\int_x^{2x} dydx = \int_0^1\int_{y/2}^{y} dxdy + \int_1^2\int_{y/2}^{1} dxdy.$$