MATH 103A December 1, 2010

WORKSHEET

1. Show that there are infinitely many homomorphism $\mathbb{Z} \to \mathbb{Z}$. Determine them all. Which ones are onto?

Let $f: \mathbb{Z} \to \mathbb{Z}$ be a homomorphism.

Then $f(k) = f(\underbrace{1 + \cdots + 1}_{k \text{ times}}) = k f(1)$ if $k > 0$

$f(-k) = -f(k) = -k f(1)$ if $k > 0$

$f(0) = 0$

Hence $f(n) = n f(1)$ for all $n \in \mathbb{Z}$

Now $f(1)$ has to be an integer $\Rightarrow$ all homomorphisms are of the form $x \to f(1)x$

So for each $a \in \mathbb{Z}$ get homomorphism $f(x) = ax$ and these are all of them. They are infinitely many, since there are infinitely many integers.

onto: a homomorphism between cyclic groups is onto iff it takes generator to generator.

So want $a = f(1)$ to be a generator of $\mathbb{Z}$ $\Rightarrow a = \pm 1$

Get 2 onto hom: $x \mapsto x$, $x \mapsto -x$

2. Determine all group homomorphisms $Z_9 \to Z_3$. Which ones are onto?

$\mathbb{Z}_9 = \{0, 1, \ldots, 8\} = \langle 1 \rangle$
$\mathbb{Z}_3 = \{0, 1, 2\}$
As before $f(x) = ax$ where $a = f(1)$
But $a = f(1) \in \mathbb{Z}_3 = \{0, 1, 2\} \Rightarrow$ 3 homomorphisms
$a = 0 \rightsquigarrow f(x) = 0$ not onto
$a = 1 \rightsquigarrow f(x) = x$ onto b/c $f(1) = \underset{\text{generates } \mathbb{Z}_3}{\underline{1}}$
$a = 2 \rightsquigarrow f(x) = 2x$ onto b/c $f(1) = \underset{\text{generates } \mathbb{Z}_3}{\underline{2}}$

Attention! well-defined b/c
$x \equiv y \pmod 9 \Rightarrow x \equiv y \pmod 3$

3. Determine all group homomorphisms $Z_9 \to Z_5$.

Here need $f(1) \in \mathbb{Z}_5 \Rightarrow |f(1)|$ is either 1 or 5
But also have $|f(1)|$ divides $|\underset{\text{in } \mathbb{Z}_9}{1}| = 9$ $\Rightarrow$
$\Rightarrow |f(1)| = 1$, hence $f(1) =$ unit in $\mathbb{Z}_5 = 0$
So $f(x) = 0$ is the only hom $f: \mathbb{Z}_9 \to \mathbb{Z}_5$

4. Assume $f : Z_m \to Z_n$ is a nontrivial homomorphism (i.e. its image contains a nonzero element). What can you say about m and n?

$\gcd(m,n) > 1$ (see 3)

5. Suppose that $f : G_1 \to G_2$ is a group homomorphism and that G_2 is abelian. Let H be a subgroup of G_1 that contains $\ker f$. Prove that H is a normal subgroup of G_1.

Want: $aha^{-1} \in H$ for all $a \in G$, $h \in H$

$f(aha^{-1}) = f(a)f(h)f(a^{-1})$ in G_2 which is abelian $\Big\} \Rightarrow$

$\Rightarrow f(aha^{-1}) = f(h) \Rightarrow f(aha^{-1}h^{-1}) = e_{G_2}$

$\Rightarrow aha^{-1}h^{-1} \in \ker f \subset H \Rightarrow aha^{-1}h^{-1} = h_1$ for some $h_1 \in H$

$\Rightarrow aha^{-1} = \underbrace{h_1}_{\in H}\underbrace{h}_{\in H} \in H$ (since H subgroup) QED.

6. Up to isomorphism, how many abelian groups of order 120 are there? Right down an example for each isomorphism class.

$120 = 2^3 \cdot 3 \cdot 5 \underset{\text{Fundam thm.}}{\Rightarrow}$ group order 8 $\oplus$ group order 3 $\oplus$ gp order 5

$\mathbb{Z}_8$ or $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$; $\mathbb{Z}_3$; $\mathbb{Z}_5$

So 3: $\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$

$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$

7. Determine the possible number of elements of order 5 in an abelian group of order 125.

$|G| = 125 \Rightarrow G \simeq \mathbb{Z}_{125}$ or $\mathbb{Z}_5 \oplus \mathbb{Z}_{25}$ or $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$

In $\mathbb{Z}_{125}$ have $\varphi(5) = \boxed{4}$ elts of order 5

In $\mathbb{Z}_{25}$ have $\varphi(5) = 4$ elts order 5

$\mathbb{Z}_5$ $\varphi(5) = 4$ elts order 5

If $(a,b) \in \mathbb{Z}_5 \oplus \mathbb{Z}_{25} \Rightarrow |(a,b)| = \gcd(|a|, |b|)$

So $|(a,b)| = 5$:

$|a| = |b| = 5$: $4 \cdot 4 = 16$ choices

$|a| = 5, |b| = 1$: $4 \cdot 1 = 4$ —"—

$|a| = 1, |b| = 5$: $1 \cdot 4 = 4$ —"—

$\boxed{24}$ —"—

Same way in $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$

$|(a,b,c)| = 5$:

$|a| = |b| = |c| = 5$: $4 \cdot 4 \cdot 4 = 64$

$|a| = |b| = 5, |c| = 1$: $4 \cdot 4$ } $\times 3 = 48$

$|a| = |c| = 5, |b| = 5$

$|a| = 5, |b| = |c| = 1$: $4 \times 3 = 12$

$\boxed{124}$