MATH 103A November 12, 2010

Exam 2

Name: ~~[illegible]~~ Solutions-yellow version

PID: ______________________

There are 6 pages and 5 questions, for a total of 100 points and 20 bonus points.

No notes, no calculators, no books.

Please turn off and put away all electronic devices.
Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. Unless otherwise stated, show all your work for full credit. To maximize credit, cross out incorrect work.

Good luck! ☺

Question	Points	Score
1	20	
2	20	
3	30	
4	10	
5	20	
Total:	100	

Question	Bonus Points	Score
3	10	
4	10	
Total:	20	

1. (20 points) Suppose that f is an automorphism of Z_9 and that $f(5) = 1$. Determine a formula for f.

Find $f(1)$. $5 \cdot 2 = 1$ in $\mathbb{Z}_9$ so $f(1) = f(2 \cdot 5) = 2f(5) = 2$.

Hence $f(k) = 2k$ by the homomorphism property.

2. (a) (5 points) How many distinct left cosets of $H = \{\varepsilon, (12)(34), (13)(24), (14)(23)\}$ are there in the group

$$A_4 = \{\varepsilon, (123), (132), (124), (142), (234), (243), (134), (143), (12)(34), (13)(24), (14)(23)\}?$$

(b) (15 points) Determine them and list them (without repetitions).

(a) $[G : H] = \frac{|G|}{|H|} = \frac{12}{4} = 3$

(b) $H = \{e, (12)(34), (13)(24), (14)(23)\}$

$(123)H = \{(123), (413), (432), (214)\}$

$(132)H = \{(132), (423), (412), (314)\}$

3. (a) (10 points) What are the possible orders of the elements of S_4?
   (b) (10 points) What are the possible orders of the elements of S_6?
   (c) (10 points) How many elements of order 2 are there in $S_3 \oplus S_3$?
   (d) (10 points (bonus)) How many elements of order 2 are there in $S_4 \oplus S_6$?

See Pink for explanation

4. (a) (10 points) Let G be a finite group, H be a subgroup of G, and K be a subgroup of H ($K \subseteq H \subseteq G$). If there are six left cosets of K in H and four left cosets of H in G, how many left cosets of K are there in G?

   (b) (10 points (bonus)) Prove your result holds even when $|G| = \infty$.

a $[G:K] = \frac{|G|}{|K|} = \frac{|G|}{|H|}\frac{|H|}{|K|} = 6 \cdot 4 = 24$

b See Pink /white

5. (20 points) Prove that a cyclic group cannot be isomorphic to $Z_n \oplus Z_n$ for any integer $n > 1$.

If $\mathbb{Z}_n \oplus \mathbb{Z}_n$ were isomorphic to a cyclic group, then it'd be isomorphic to $\mathbb{Z}_{n^2}$. However, $\mathbb{Z}_n \oplus \mathbb{Z}_n$ has two distinct subgroups of order n. But $\mathbb{Z}_{n^2}$ only has one.