MATH 103A

November 12, 2010

Exam 2

Name: SOLUTIONS - white version

PID: ______________________

There are 6 pages and 5 questions, for a total of 100 points and 20 bonus points.

No notes, no calculators, no books.

Please turn off and put away all electronic devices.
Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. Unless otherwise stated, show all your work for full credit. To maximize credit, cross out incorrect work.

Good luck! ☺

Question	Points	Score
1	20	
2	20	
3	30	
4	10	
5	20	
Total:	100	

Question	Bonus Points	Score
3	10	
4	10	
Total:	20	

1. (20 points) Suppose that f is an automorphism of Z_8 and that $f(3) = 1$. Determine a formula for f.

Since f is an automorphism of $\mathbb{Z}_8$, it is of the form $f(x) = r\,x$ for some $r \in U(8)$

But $f(3) = 1 \Rightarrow 3r \equiv 1 \pmod{8}$

$r \in U(8) = \{1, 3, 5, 7\}$

Hence $r = 3$ and $\boxed{f(x) = 3x}$

2. (a) (5 points) How many distinct left cosets of $H = \{\varepsilon, (234), (243)\}$ are there in the group

$$A_4 = \{\varepsilon, (123), (132), (124), (142), (234), (243), (134), (143), (12)(34), (13)(24), (14)(23)\}?$$

(b) (15 points) Determine them and list them (without repetitions).

(a) $\left.\begin{array}{l} |H| = 3 \\ |A_4| = 12 \end{array}\right| \Rightarrow \frac{12}{3} = 4$ left cosets of H in A_4

(b) $\varepsilon H = H = \{\varepsilon, (234), (243)\} = (234)H = (243)H$

$(123)H = \{(123), (12)(34), (124)\} = (12)(34)H = (124)H$

$(132)H = \{(132), (134), (13)(24)\} = (134)H = (13)(24)H$

$(142)H = \{(142), (14)(23), (143)\} = (14)(23)H = (143)H.$

3. (a) (10 points) What are the possible orders of the elements of S_5?
   (b) (10 points) What are the possible orders of the elements of S_4?
   (c) (10 points) How many elements of order 2 are there in $S_3 \oplus S_3$?
   (d) (10 points (bonus)) How many elements of order 2 are there in $S_5 \oplus S_4$?

(a) Looking at disjoint cycles decomposition, we see that possible orders are

(length 5) order = 5
(length 4) (length 1) order = lcm(4,1) = 4
(length 3) (length 2) order = lcm(3,2) = 6
(length 3)(length 1) (length 1) order = lcm(3,1,1) = 3
(length 2) (length 2) (length 1) order = lcm(2,2,1) = 2
$(l=2)\ (l=1)\ (l=1)\ (l=1)$ order = lcm(2,1,1,1) = 2
unit order = 1

possible orders: 1, 2, 3, 4, 5, 6

(b) Same argument

$(l=4)$ order = 4
$(l=3)(l=1)$ order = lcm(3,1) = 3
$(l=2)(l=2)$ order = lcm(2,2) = 2
$(l=2)(l=1)(l=1)$ order = lcm(2,1,1) = 2
unit order = 1

possible orders = 1, 2, 3, 4

(c) $|(\sigma,\tau)| = \operatorname{lcm}(|\sigma|, |\tau|)$

So $|(\sigma,\tau)| = 2$ in 3 cases

$|\sigma| = 2, \ |\tau| = 2$

$|\sigma| = 2, \ |\tau| = 1$

$|\tau| = 2, \ |\sigma| = 1$

The only elements of order 2 in S_3 are permutations, and there are 3 of them: (12), (13) and (23)

So

$\lvert\sigma\rvert = 2, \lvert\tau\rvert = 2$:	$3 \cdot 3 = 9$	possibilities
$\lvert\sigma\rvert = 2, \lvert\tau\rvert = 1$:	$3 \cdot 1 = 3$	— " —
$\lvert\sigma\rvert = 1, \lvert\tau\rvert = 2$:	$1 \cdot 3 = 3$	— " —
	$\boxed{15}$	— " —

(d) The elements of order 2 in S_5 are

$(\ell=2)(\ell=2)$: $5 \cdot 3 = 15$ elements

$(\ell=2)$: $\binom{5}{2} = \frac{4 \cdot 5}{2} = 10$ elements

} 25 elements

In S_4: $(\ell=2)(\ell=2)$: 3

$(\ell=2)$: $6 = \binom{4}{2}$ elements

} 9 elements

So $|(\sigma,\tau)| = 2$, $\sigma \in S_5$, $\tau \in S_4$

$|\sigma| = 2, |\tau| = 2$: $9 \cdot 25 = 225$ poss.

$|\sigma| = 2, |\tau| = 1$: $25 \cdot 1 = 25$

$|\sigma| = 1, |\tau| = 2$: 9 possibilities

Hence $\boxed{259}$ elements.

4. (a) (10 points) Let G be a finite group, H be a subgroup of G, and K be a subgroup of H ($K \subseteq H \subseteq G$). If there are seven left cosets of K in H and six left cosets of H in G, how many left cosets of K are there in G?

(b) (10 points (bonus)) Prove your result holds even when $|G| = \infty$.

(a) 7 left cosest of K in $H \Rightarrow \frac{|H|}{|K|} = 7$

6 — " — H in $G \Rightarrow \frac{|G|}{|H|} = 6$

left ~~cosets of~~ K in G $= \frac{|G|}{|K|} = \frac{|G|}{|H|} \cdot \frac{|H|}{|K|} = 7 \cdot 6 = \boxed{42}$

(b) The left cosets of [redacted] H in G are

$a_1H, a_2H, \dots, a_6H$ for some $a_1, \dots, a_6 \in G$

The left cosets of K in H are $b_1K, \dots, b_7K$ for some $b_1, \dots, b_7 \in H$

Claim The left cosets of K in G are $\left\{a_i b_j K \;;\; \begin{matrix} 1 \le i \le 6 \\ 1 \le j \le 7 \end{matrix}\right\}$

In particular, there are 42 such cosets.

Proof of claim Since $a_1H, \dots, a_6H$ are the cosets of H in G, it follows that $G = \bigsqcup_{i=1}^{6} a_iH$ (disjoint union)

For the same reason $H = \bigsqcup_{j=1}^{7} b_jK$

So $G = \bigsqcup_{i=1}^{6} a_i \left(\bigsqcup_{j=1}^{7} b_jK \right) = \bigsqcup_{i=1}^{6} \bigsqcup_{j=1}^{7} a_ib_jK \Rightarrow$ these are all the left cosets of K in G. But there might be some overlap, and we have to rule out this possibility.

Assume $a_ib_jK \cap a_mb_\ell K \neq \emptyset \Rightarrow \exists k_1, k_2 \in K$ s.t. $a_ib_jk_1 = a_mb_\ell k_2 \Rightarrow a_m^{-1}a_i \underbrace{b_jk_1}_{\in H} = \underbrace{b_\ell k_2}_{\in H} \Rightarrow a_m^{-1}a_i \in H$

$\Rightarrow a_mH = a_iH \Rightarrow a_m = a_i$ (b/c they were distinct cosets) $\Rightarrow m = i$

So $b_jk_1 = b_\ell k_2 \Rightarrow b_j \in b_\ell K \Rightarrow b_j = b_\ell$ (— " —) $\Rightarrow j = \ell$

Hence $a_ib_jK = a_mb_\ell K$ and $i = m, j = \ell$. So there is no overlap.

QED

5. (20 points) Prove that, for every prime p, $Z_p \oplus U(p)$ is cyclic. You can use the fact that $U(p)$ is cyclic for any prime p.

$U(p)$ cyclic, $\mathbb{Z}_p$ cyclic

So $\mathbb{Z}_p \oplus U(p)$ cyclic $\Leftrightarrow \gcd(|\mathbb{Z}_p|, |U(p)|) = 1$

But $|\mathbb{Z}_p| = p$

$|U(p)| = p-1$

Let $d = \gcd(p, p-1) \Rightarrow \left.\begin{array}{l} d \mid p \\ d \mid p-1 \end{array}\right\} \ominus$

$d \mid p-(p-1) = 1 \Rightarrow d = 1$

Hence $\mathbb{Z}_p \oplus U(p)$ cyclic