MATH 103A

November 12, 2010

Exam 2

Name: SOLUTIONS -pink version

PID: ______________________

There are 6 pages and 5 questions, for a total of 100 points and 20 bonus points.

No notes, no calculators, no books.

Please turn off and put away all electronic devices.
Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. Unless otherwise stated, show all your work for full credit. To maximize credit, cross out incorrect work.

Good luck! ☺

Question	Points	Score
1	20	
2	20	
3	30	
4	10	
5	20	
Total:	100	

Question	Bonus Points	Score
3	10	
4	10	
Total:	20	

1. (20 points) Suppose that f is an automorphism of Z_9 and that $f(4) = 1$. Determine a formula for f.

All automorphisms of $\mathbb{Z}_9$ are of the form

$f(x) = rx$ for some $r \in U(9) = \{1, 2, 4, 5, 7, 8\}$

Want $f(4) = 1 \iff 4r \equiv 1 \pmod 9$

So $r = 7$

Hence $\boxed{f(x) = 7x}$

2. (a) (5 points) How many distinct left cosets of $H = \{\varepsilon, (134), (143)\}$ are there in the group

$$A_4 = \{\varepsilon, (123), (132), (124), (142), (234), (243), (134), (143), (12)(34), (13)(24), (14)(23)\}?$$

(b) (15 points) Determine them and list them (without repetitions).

(a) $[G:H] = \dfrac{|G|}{|H|} = \dfrac{12}{3} = 4.$

(b) $H = \{e, (134), (143)\}$

$(123)H = \{(123), (423), (32)(14)\}$

$(132)H = \{(132), (43)(12), (3)(421)\}$

$(124)H = \{(124), (42)(13), (324)\}$

3. (a) (10 points) What are the possible orders of the elements of S_6?
   (b) (10 points) What are the possible orders of the elements of S_4?
   (c) (10 points) How many elements of order 2 are there in $S_3 \oplus S_3$?
   (d) (10 points (bonus)) How many elements of order 2 are there in $S_6 \oplus S_4$?

(a) Consider the possible cycle decompositions

$|(6\text{-cycle})| = 6$

$|(5\text{-cycle})| = 5$

$|(4\text{-cycle})| = 4$

$|(4\text{-cycle})(2\text{-cycle})| = 4$

$|(3\text{-cycle})| = 3$

$|(3\text{-cycle})(2\text{-cycle})| = 6$

$|(3\text{-cycle})(3\text{-cycle})| = 3$

$|e| = 1$

$|(2\text{-cycle})| = 2$

$|(2\text{-cycle})(2\text{-cycle})| = 2$

$|(2\text{-cycle})(2\text{-cycle})(2\text{-cycle})| = 2$

So 1, 2, 3, 4, 5, 6 are the possible orders.

(b) Do the same. I have already listed all the possible cycles for S_4 in the ~~|(4-cycle)|~~ list above. The orders can be 1, 2, 3, or 4.

(c) If $(a,b) \in S_3 \oplus S_3$ has $|(a,b)| = \mathrm{lcm}(|a|,|b|) = 2$ then either $|a| = |b| = 2$ or one is order 2 and the other is order 1.

$|a| = 2$

Then $|b| = 2$ or 1. There are 3 such a and 4 such b. So 12 such (a,b).

$|a| = 1$

Then $|b| = 2$. There is 1 a and 3 b so 3 such (a,b).

In total, there are 15 such (a,b).

(d)

elements of order 2 in S_6 :

(2-cycle)(2-cycle)(2-cycle) : $\binom{6}{2} = 15$ possibilities

(2-cycle)(2-cycle) : $\binom{6}{2} \cdot 3 = 45$ — " —

(2-cycle) : $\binom{6}{2} = 15$ — " —

75 elements

elements of order 2 in S_4 :

(2-cycle)(2-cycle) : 3

(2 cycle) : $\binom{4}{2} = 6$

9 elements

So there are $75 \cdot 9 + 75 + 9 = \boxed{759}$ elements of order 2 in $S_6 \oplus S_4$

4. (a) (10 points) Let G be a finite group, H be a subgroup of G, and K be a subgroup of H ($K \subseteq H \subseteq G$). If there are three left cosets of K in H and eight left cosets of H in G, how many left cosets of K are there in G?

   (b) (10 points (bonus)) Prove your result holds even when $|G| = \infty$.

(a) $8 = \frac{|G|}{|H|}$ And $3 = \frac{|H|}{|K|}$.

Hence $\frac{|G|}{|K|} = \frac{|G|}{|H|}\frac{|H|}{|K|} = 8 \cdot 3 = 24$

(b) Let $\{a_i H \mid i = 1 \ldots 8\}$ be the cosets of H in G. Let $\{b_j K \mid j = 1 \ldots 3\}$ be the cosets of K in H. Then

$G = \bigcup_{i=1}^{8} a_i H = \bigcup_{i=1}^{8} a_i \left(\bigcup_{j=1}^{3} b_j K \right) = \bigcup_{i,j} a_i b_j K$. So we have written G as a ~~disjoint~~ union of cosets of K. Now we show they are disjoint. If $a_i b_j K = a_\ell b_n K$ then $a_i b_j = a_\ell b_n k$ for some $k \in K$. Since $K \leq H$. We then have $a_i = a_\ell b_n k b_j^{-1}$ where $b_n k b_j^{-1} \in H$. So $a_i H = a_\ell H$. Hence $i = \ell$. And then $b_j = b_n k$ so $b_j K = b_n K$ and we find $j = n$ too. Thus all these cosets are mutally disjoint, and are thus <u>all</u> the cosets of K.

So $[G:K] = 24$.

5. (20 points) Prove or disprove:

For any integer n, $Z_n \oplus Z_{n+1}$ is isomorphic to a cyclic group.

$\mathbb{Z}_n$ and $\mathbb{Z}_{n+1}$ are both cyclic

So $\mathbb{Z}_n \oplus \mathbb{Z}_{n+1}$ is cyclic $\iff \gcd(|\mathbb{Z}_n|, |\mathbb{Z}_{n+1}|) = 1$
$\iff \gcd(n, n+1) = 1$

this is true b/c any number d that divides both n and (n+1) will have to divide their difference

$$\left.\begin{array}{l} d \mid n \\ d \mid n+1 \end{array}\right| \Rightarrow d \mid (n+1) - n = 1, \text{ so } d = 1$$