MATH 103A October 20, 2010

Exam 1

Name: SOLUTIONS - yellow version

PID: ______________________

There are 7 pages and 6 questions, for a total of 120 points.

No notes, no calculators, no books.

Please turn off and put away all electronic devices.
Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. Unless otherwise stated, show all your work for full credit. To maximize credit, cross out incorrect work.

Good luck! ☺

Question	Points	Score
1	15	
2	20	
3	20	
4	20	
5	35	
6	10	
Total:	120	

1. (15 points) List all the cyclic subgroups of $U(16)$.

$U(16) = \{1, 3, 5, 7, 9, 11, 13, 15\}$

$\langle 1 \rangle = \{1\}$

$\langle 3 \rangle = \{1, 3, 9, 11\}$

$\langle 5 \rangle = \{1, 5, 9, 13\}$

$\langle 9 \rangle = \{1, 9\}$

$\langle 11 \rangle = \{1, 11, 9, 3\} = \langle 3 \rangle$

$\langle 13 \rangle = \{1, 13, 9, 5\} = \langle 5 \rangle$

$\langle 15 \rangle = \{1, 15\}$

Cyclic subgroups

$\{1\} = \langle 1 \rangle$

$\{1, 3, 9, 11\} = \langle 3 \rangle = \langle 11 \rangle$

$\{1, 5, 9, 13\} = \langle 5 \rangle = \langle 13 \rangle$

$\{1, 9\} = \langle 9 \rangle$

$\{1, 15\} = \langle 15 \rangle$

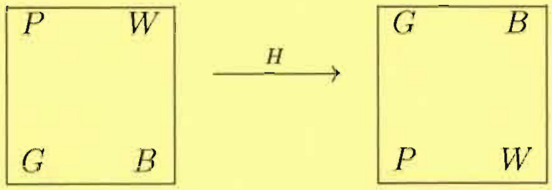
2. (a) (10 points) How many elements of order 4 does ($Z_{12}, +$ (mod 12)) have? Justify your answer.

   (b) (10 points) How many elements of order 4 does Z_{2010} have? Justify your answer.

(a) $\left.\begin{array}{l} 4 \mid 12 = |\mathbb{Z}_{12}| \\ \mathbb{Z}_{12} \text{ is cyclic} \end{array}\right\} \Rightarrow$ it has $\varphi(4) = 2$ elements of order 4

(b) $\mathbb{Z}_{2010}$ is cyclic, but $4 \nmid 2010$.
Hence there are <u>no</u> elements of order 4 in $\mathbb{Z}_{2010}$.

3. (20 points) Compute the centralizer in D_4 of the flip across the horizontal axis.



It might help (**but you don't have to use this if you know a simpler way to solve the problem**) to know that the matrix of a [redacted]rclockwise rotation by angle θ is $R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, and the matrix of a flip across a line that makes an angle α with the positive x-axis is $S_\alpha = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$.

Let σ denote clockwise rotation and τ denote the flip about the (horizontal) axis. Then we have that $\sigma^4 = \tau^2 = \sigma\tau\sigma\tau = e$, And $D_4 = \{e, \sigma, \sigma^2, \sigma^3, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\}$.

$\sigma\tau\sigma\tau = e$ gives us that $\sigma\tau\sigma = \tau$ And $\tau\sigma = \sigma^3\tau$. We'll use these relations to compute $C(\{t\})$.

$e\tau = \tau e = \tau$ so $e \in C(\{t\})$

$\tau\tau = \tau\tau = e$ so $\tau \in C(\{t\})$

$\tau\sigma^2 = \sigma^3\tau\sigma = \sigma^6\tau = \sigma^2\tau$ so $\sigma^2 \in C(\{t\})$.

$\sigma^3\tau = \tau\sigma \neq \tau\sigma^3$ so $\sigma^3 \notin C(\{t\})$.

$\sigma\tau = \tau\sigma^3 \neq \tau\sigma$ so $\sigma \notin C(\{t\})$

$C(\{t\})$ is a subgroup so $\tau, \sigma^2 \in C(\{t\})$ implies that $\sigma^2\tau \in C(\{t\})$.

~~Similarly~~, Finally, $(\sigma\tau)\tau = \sigma$ And $\tau(\sigma\tau) = \sigma^3\tau\tau = \sigma^3$ so $\sigma\tau \notin C(\{t\})$. $(\sigma^3\tau)\tau = \sigma^3$ And $\tau(\sigma^3\tau) = \sigma\tau$ so $\sigma^3\tau \notin C(\{t\})$ either. Thus $C(\{t\}) = \{e, \tau, \sigma^2$, And $\sigma^2\tau\}$.

4. (20 points) Suppose that $(G, \cdot)$ is a group and that $x^2 = e$ for all elements $x \in G$. Prove that G is abelian.

Let $a, b \in G$. Then $(ab)^2 = e$ and $a^2 b^2 = e$. So we have $(ab)^2 = a^2 b^2$. Then

$$abab = aabb$$
$$bab = abb$$
$$ba = ab$$

Since $ba = ab$ for any $a, b \in G$ G is abelian.

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5. Let $G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} ; a \in \mathbb{R}, a \neq 0 \right\}$.

(a) (10 points) Find an element of G that acts like the identity element in G with respect to matrix multiplication (i.e. find $X \in G$ such that $AX = XA = A$ for all $A \in G$.)

(b) (10 points) Show that G is closed under matrix multiplication (i.e. show that $AB \in G$ for any matrices $A \in G$ and $B \in G$.)

(c) (15 points) Show that (G, matrix multiplication) is a group. You can assume associativity, since we know that matrix multiplication is associative in general.

see blue version

6. (10 points) Decompose into disjoint cycles the permutation

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 2 & 3 & 9 & 5 & 1 & 8 & 6 \end{bmatrix}.$$

$(17)(243)(596)(8)$