MATH 103A - Makeup problems

to be handed in October 29, 2010

Justify all your answers (even when the problem only asks you to compute or list something). You will get points for both clarity of exposition, as well as the argument itself. Be as clear as possible. Due to time constraints, if I have trouble understanding what you mean, you will get no credit.

1. Let $G = \{a + b\sqrt{2}; a, b \in \mathbb{Q}\}$ and define on it the operation $g_1 * g_2 = g_1 + g_2 + \sqrt{2}$ for all $g_1, g_2 \in G$. Show that G is an abelian group. Indicate the unit and the inverse of $a + b\sqrt{2}$ for all $a, b \in \mathbb{Q}$. Is G cyclic? What is the cyclic subgroup generated by $1 - 3\sqrt{2}$?

2. Let $G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix}; a \in \mathbb{R}, a \neq 0 \right\}$. Show that G is a group under the usual matrix multiplication. Indicate the unit and give the formula for the inverse of an element of G. Is G cyclic? Is it abelian?

3. Let $G = \left\{ \begin{bmatrix} 1-x & 0 & x \\ 0 & 0 & 0 \\ x & 0 & 1-x \end{bmatrix}; a \in \mathbb{R}, a \neq \frac{1}{2} \right\}$. Show that G is a group under the usual matrix multiplication. Indicate the unit and give the formula for the inverse of an element of G. Is G cyclic? Is it abelian?

4. Let $G = \{e, a, b, c\}$ be a set with four elements. Suppose $(G, \cdot)$ is a group with unit e and $a^2 = b^2 = c^2 = e$. Complete the Cayley table of G. Is G abelian? Is G cyclic?

5. Suppose that $(G, \cdot)$ is a group and that $x^2 = e$ for all elements $x \in G$. Prove that G is abelian.

6. Suppose that $(G, \cdot)$ is a group and that it has an element $a \in G$ with the property that $ax^3a = x$ for all elements $x \in G$. Prove that G is abelian.

7. Suppose $(G, \cdot)$ is a group and $a, b \in G$ have the property that $b^6 = e$ and $ab = b^4a$. Suppose further that $b \neq e$. Show that the order of b is 3 and $ab = ba$.

8. Suppose $(G, \cdot)$ is a group and $a, b \in G$ have the property that $a^2 = b^2 = (ab)^2$. What are the possible orders of $|a|$ and $|b|$?

9. Let $\mathrm{GL}(2, Z_5)$ the group of invertible 2×2 matrices with entries taken from Z_5 and where all arithmetic is done modulo 5. The operation is matrix multiplication. Compute the inverse of $A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$. Compute the centralizer of A in G.

10. List all the generators for the subgroup of order 9 in Z_{252}. Justify your answer.

11. List all the generators for the subgroup of order 12 in Z_{252}. Justify your answer.

12. List all the generators for the subgroup of order 7 in Z_{252}. Justify your answer.

13. Find $A = \{a \in U(9); a^2 = 1\}$ and $B = \{b^2; b \in U(9)\}$.

14. Find $A = \{a \in U(7); a^2 = 1\}$ and $B = \{b^2; b \in U(7)\}$.

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