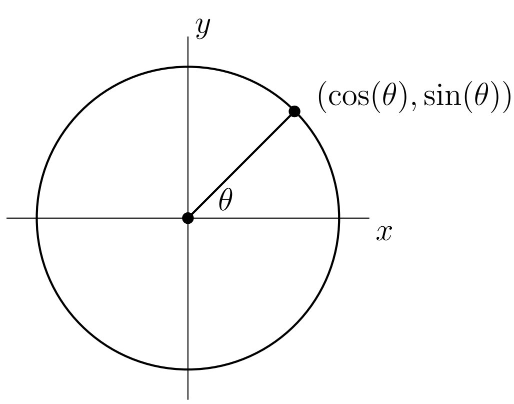
Trigonometry Outline

Introduction

Knowledge of the content of this outline is essential to perform well in calculus. The reader is urged to study each of the three parts of the outline.

Part I contains the definitions and theorems that are needed to understand the content of Part II.

Part II has a page for each of the six trigonometric functions. On each page, a trigonometric function and its inverse are defined, and the properties of that function are listed. The reader is urged to understand why each property holds rather than simply memorize the property. For example, the cosine and sine functions are defined so that $x = \cos(\theta)$ and $y = \sin(\theta)$, where (x, y) is the point on the unit circle $x^2+y^2 = 1$ where the terminal side of an angle of measure θ (with initial side the positive x-axis) intersects the unit circle.



It should be clear from the definition that both x and y take on every value in the interval $[-1, 1]$, and consequently the range of both the cosine and sine functions is $[-1, 1]$. It should also be clear that

$$\cos^2(\theta) + \sin^2(\theta) = 1,$$

which is the **Pythagorean identity**. So the reader should remember that the Pythagorean identity follows from the geometric properties of the unit circle and how the sine and cosine functions are defined in relation to the unit circle.

As another example, consider the tangent and secant functions:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \quad \text{and} \quad \sec(\theta) = \frac{1}{\cos(\theta)}.$$

Since division by zero is not defined, we cannot have $\cos(\theta) = 0$, and so we cannot have values of θ that will place the point (x, y) on the y-axis. Therefore, θ cannot have the values $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \cdots$. Both tangent and secant have the same domain, and that domain is the set of all real numbers except odd multiples of $\pi/2$. In set notation, this is written:

$$\left\{\theta \in \mathbb{R} : \theta \neq \frac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z}\right\}.$$

As a final example, note that the angles θ and $-\theta$ have terminal sides that intersect the unit circle at (x, y) and $(x, -y)$, respectively. It follows that $\cos(-\theta) = \cos(\theta)$, since both of these quantities equal x, and $\sin(-\theta) = -\sin(\theta)$, since both of these quantities equal $-y$. Consequently, cosine is an even function and sine is an odd function.

Part III is a list of the basic trig identities. The reader is encouraged to understand where these identities come from, and how to derive them, rather than simply memorize them all.

Part I: Definitions and Theorems

A **function** is a rule that assigns to each element of a set, called the **domain**, one and only one element of another set, called the **codomain**. If f is the name of the function, D is the domain, and C is the codomain, then for each $x \in D$, the unique member of C corresponding to x is denoted $f(x)$, and is called the **image** of x. If we write $y = f(x)$, then x is called the **independent variable** and y is called the **dependent variable**. The set of all images is called the **range** of the function.

In these notes, unless otherwise specified, the domain and codomain of a function will always be the set of real numbers.

If f is a function, the set of all points with coordinates (x, y), where $y = f(x)$, is called the **graph** of the function f. In set notation, this is written

$$\left\{(x, f(x)) : x \in D\right\},$$

where D is the domain of f.

A function f is called **even** if $f(-x) = f(x)$ for all x in the domain of f, and it is called **odd** if $f(-x) = -f(x)$ for all x in the domain of f.

Theorem 1. *Let f and g be functions, and let $S = f + g$, $D = f - g$, $P = f \cdot g$, and $Q = f/g$.*

(a) If f and g are both even, then S, D, P, and Q are all even.
(b) If f and g are both odd, then S and D are odd, but P and Q are even.
(c) If f is even and g is odd, or visa versa, then P and Q are odd.

Theorem 2. *Let f be a function.*

(a) If f is even, then its graph is symmetric with respect to the y-axis.
(b) If f is odd, then its graph is symmetric with respect to the origin.

A function f is called **periodic** if there is a number $p \neq 0$ such that $f(x+p) = f(x)$ for all x in the domain of f. In this case, f is said to be periodic with period p, and p is called a period of f.

Theorem 3. *If f is a periodic function with period p, then*

$$f(x) = f(x+p) = f(x+2p) = f(x+3p) = \cdots = f(x+np) = \cdots ,$$

for all $n \in \mathbb{N}$. Furthermore, if p is a period of f, then so is kp for all $k \in \mathbb{N}$.

If a function is periodic, then the smallest positive period is called the **fundamental period** of the function.

A function f is said to be a **one-to-one** (or **injective**) function provided that $f(x_1) = f(x_2)$ implies $x_1 = x_2$, whenever x_1 and x_2 are in the domain of f. An equivalent statement comes from the contrapositive: f is a one-to-one function provided that whenever x_1 and x_2 are two distinct elements in the domain of f, it follows that $f(x_1) \neq f(x_2)$.

Theorem 4 (Horizontal Line Test)**.** *A function f is one-to-one if and only if each horizontal line intersects the graph of f in at most one point.*

Let f be a one-to-one function with domain D and range R. The **inverse** of f is denoted f^{-1} and is the function with domain R and range D such that

$$y = f(x) \quad \text{if and only if} \quad f^{-1}(y) = x.$$

Therefore, $f^{-1}\big(f(x)\big) = x$ for all $x \in D$ and $f\big(f^{-1}(x)\big) = x$ for all $x \in R$.

Theorem 5. *Let f be a one-to-one function. The graph of f^{-1} is obtained by reflecting the graph of f through the line $y = x$.*

Let f be a function with domain D. If A is a subset of D, then the **restriction of f to A** is the function g with domain A defined by $g(x) = f(x)$ for all $x \in A$. The restriction of f to A is denoted $f|_A$.

Remark. If f is not a one-to-one function, then it does not have an inverse. Sometimes, however, there will be a subset A of the domain of f such that $f|_A$ is one-to-one, and so $f|_A$ will have an inverse.

Let f be a function and let a be in the domain of f. We say that f is **continuous at** a provided that $\lim_{x \to a} f(x)$ exists and

$$\lim_{x \to a} f(x) = f(a).$$

A function is called a **continuous function** if it is continuous at every point in its domain. The **derivative** of f at a is defined to be

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists. If the derivative of f exists at a, then the function f is said to be **differentiable at** a. A function that is differentiable at every point in its domain is called a **differentiable function**.

Theorem 6. *If a function is differentiable at a, then it is continuous at a.*

Remark. The converse to the above theorem is not true. The function $f(x) = |x|$ is continuous at $x = 0$, but it is *not* differentiable there.

Notation. If f is a differentiable function, and $y = f(x)$, then the derivative of f at a is denoted by each of the following symbols:

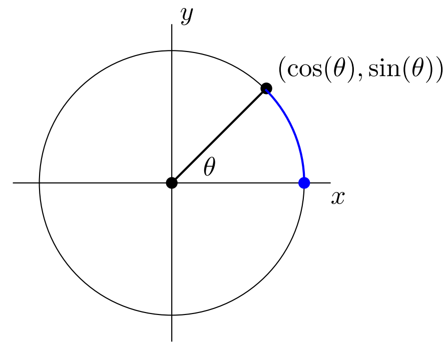
$$f'(a),\ \frac{df}{dx}(a),\ D_a\, f(x),\ D_a\, y,\ \left.\frac{df}{dx}\right|_{x=a},\ \left.\frac{dy}{dx}\right|_{x=a}.$$

Part II: Summary of trigonometric functions

The following pages contain summaries of the six trigonometric functions.

The cosine function

Definition: Consider an angle with vertex at the origin, initial side on the positive x-axis, and having angle measure $|\theta|$. Form the angle by rotating the terminal side counterclockwise if $\theta > 0$, and clockwise if $\theta < 0$. Let (x, y) be the coordinates of the point where the terminal side intersects the unit circle $x^2 + y^2 = 1$. We define the cosine function by setting $x = \cos(\theta)$.

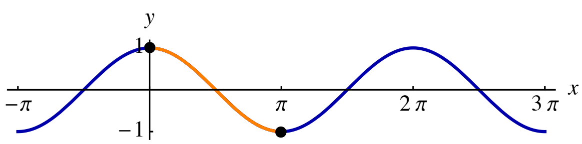


$x = \cos(\theta)$
Intercepted arc has length $|\theta|$.

Properties:

(a) The domain is $(-\infty, \infty)$.
(b) The range is $[-1, 1]$.
(c) It is an even function.
(d) It is periodic with fundamental period 2π.
(e) It is differentiable and $\dfrac{d}{dx}\cos(x) = -\sin(x)$.

Graph: The cosine function is one-to-one on the interval $[0, \pi]$.



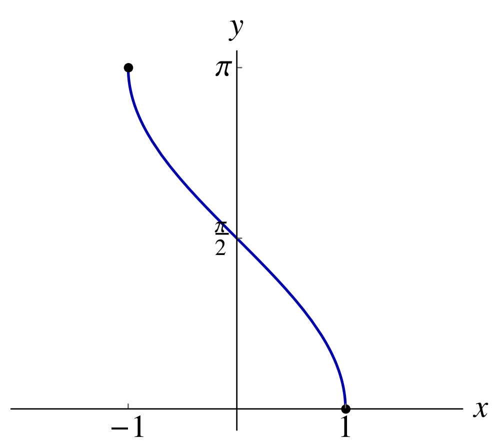
Inverse: The inverse of $f(x) = \cos(x)$ on the interval $[0, \pi]$ is $f^{-1}(x) = \arccos(x)$.

(a) The domain of arccos is $[-1, 1]$ and the range is $[0, \pi]$.

(b) $\arccos(\cos(x)) = x$ for all $0 \le x \le \pi$.

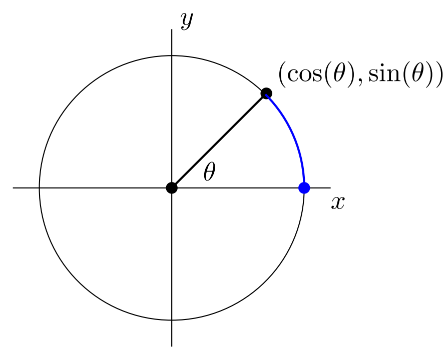
(c) $\cos(\arccos(x)) = x$ for all $-1 \le x \le 1$.

(d) $\dfrac{d}{dx}\arccos(x) = -\dfrac{1}{\sqrt{1 - x^2}}$.



The sine function

Definition: Consider an angle with vertex at the origin, initial side on the positive x-axis, and having angle measure $|\theta|$. Form the angle by rotating the terminal side counterclockwise if $\theta > 0$, and clockwise if $\theta < 0$. Let (x, y) be the coordinates of the point where the terminal side intersects the unit circle $x^2 + y^2 = 1$. We define the sine function by setting $y = \sin(\theta)$.

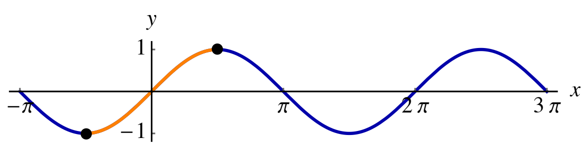


$x = \sin(\theta)$
Intercepted arc has length $|\theta|$.

Properties:

(a) The domain is $(-\infty, \infty)$.
(b) The range is $[-1, 1]$.
(c) It is an odd function.
(d) It is periodic with fundamental period 2π.
(e) It is differentiable and $\dfrac{d}{dx}\sin(x) = \cos(x)$.

Graph: The sine function is one-to-one on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



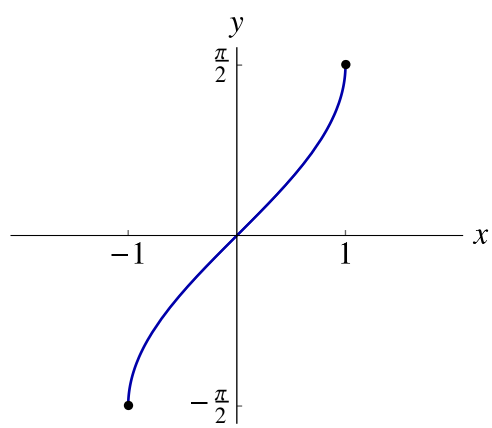
Inverse: The inverse of $f(x) = \sin(x)$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is $f^{-1}(x) = \arcsin(x)$.

(a) The domain of arcsine is $[-1, 1]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) $\arcsin(\sin(x)) = x$ for all $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

(c) $\sin(\arcsin(x)) = x$ for all $-1 \le x \le 1$.

(d) $\dfrac{d}{dx}\arcsin(x) = \dfrac{1}{\sqrt{1 - x^2}}$.

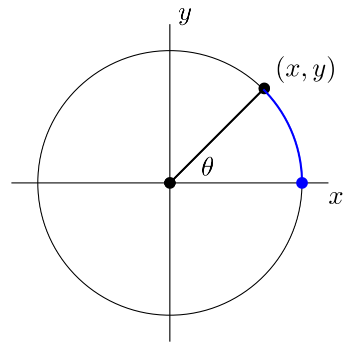


The tangent function

Definition: We define the tangent function by

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.$$

In terms of the unit circle, form the angle with vertex at the origin, initial side on the positive x-axis, and having angle measure $|\theta|$, rotating the terminal side counterclockwise if $\theta > 0$, and clockwise if $\theta < 0$. Let (x, y) be the coordinates of the point where the terminal side intersects the unit circle $x^2 + y^2 = 1$. Then $\tan(\theta) = \frac{y}{x}$.

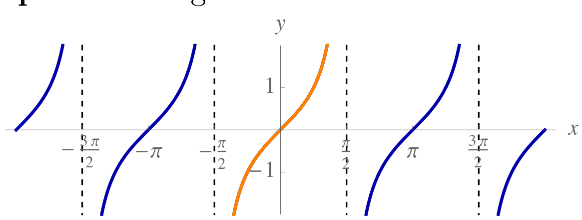


$\tan(\theta) = \frac{y}{x}$

Properties:

(a) The domain is the set of all real numbers except the odd multiples of $\pi/2$; that is:
$$\left\{ x : x \neq \tfrac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z} \right\}.$$
(b) The range is $(-\infty, \infty)$.
(c) It is an odd function.
(d) It is periodic with fundamental period π.
(e) It is differentiable and $\frac{d}{dx}\tan(x) = [\sec(x)]^2$.

Graph: The tangent function is one-to-one on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



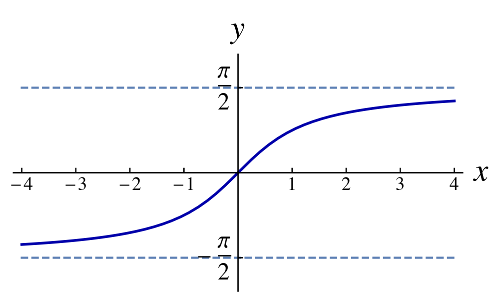
Inverse: The inverse of $f(x) = \tan(x)$ on the interval $[-\pi/2, \pi/2]$ is $f^{-1}(x) = \arctan(x)$.

(a) The domain of arctan is $(-\infty, \infty)$ and the range is $[-\pi/2, \pi/2]$.

(b) $\arctan(\tan(x)) = x$ for all $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

(c) $\tan(\arctan(x)) = x$ for all $-\infty < x < \infty$

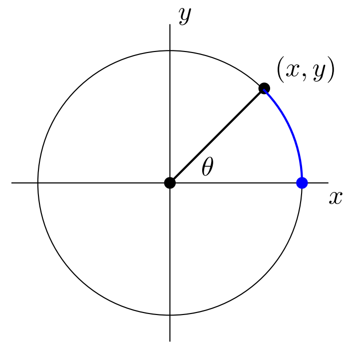
(d) $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$



The secant function

Definition: We define the secant function by

$$\sec(\theta) = \frac{1}{\cos(\theta)}.$$



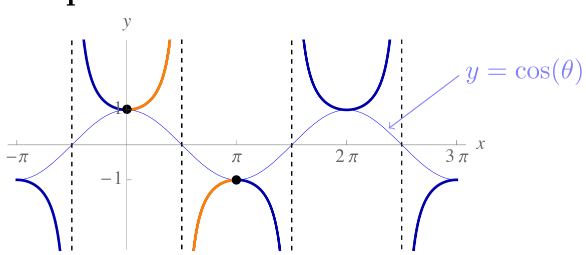
$\sec(\theta) = \frac{1}{x}$

In terms of the unit circle, form the angle with vertex at the origin, initial side on the positive x-axis, and having angle measure $|\theta|$, rotating the terminal side counterclockwise if $\theta > 0$, and clockwise if $\theta < 0$. Let (x, y) be the coordinates of the point where the terminal side intersects the unit circle $x^2 + y^2 = 1$. Then $\sec(\theta) = \frac{1}{x}$.

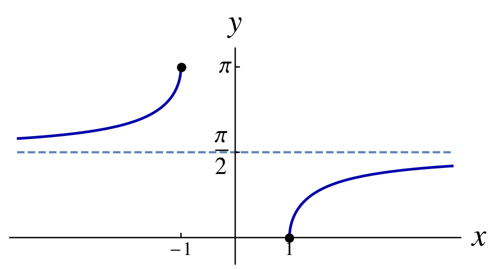
Properties:

(a) The domain is the set of all real numbers except the odd multiples of $\pi/2$; that is:
$$\left\{x : x \neq \tfrac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z}\right\}.$$
(b) The range is $(-\infty, -1] \cup [1, \infty)$.
(c) It is an even function.
(d) It is periodic with fundamental period 2π.
(e) It is differentiable and $\frac{d}{dx}\sec(x) = \tan(x)\sec(x)$.

Graph: The secant function is one-to-one on the interval $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.



Inverse: The inverse of $f(x) = \sec(x)$ on the interval $[0, \pi/2) \cup (\pi/2, \pi]$ is $f^{-1}(x) = \operatorname{arcsec}(x)$.



(a) The domain of arcsec is $(-\infty, -1] \cup [1, \infty)$ and the range is $[0, \pi/2) \cup (\pi/2, \pi]$

(b) $\operatorname{arcsec}(\sec(x)) = x$ for $x \in [0, \pi/2) \cup (\pi/2, \pi]$

(c) $\sec(\operatorname{arcsec}(x)) = x$ for $x \in (-\infty, -1] \cup [1, \infty)$

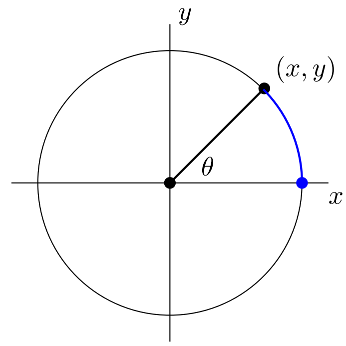
(d) $\frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{x\sqrt{x^2-1}}$

The cosecant function

Definition: We define the cosecant function by

$$\csc(\theta) = \frac{1}{\sin(\theta)}.$$

In terms of the unit circle, form the angle with vertex at the origin, initial side on the positive x-axis, and having angle measure $|\theta|$, rotating the terminal side counterclockwise if $\theta > 0$, and clockwise if $\theta < 0$. Let (x, y) be the coordinates of the point where the terminal side intersects the unit circle $x^2 + y^2 = 1$. Then $\csc(\theta) = \frac{1}{y}$.

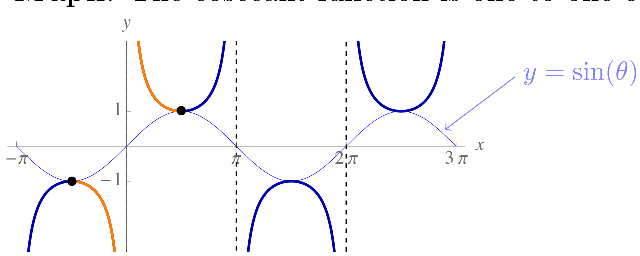


$\csc(\theta) = \frac{1}{y}$

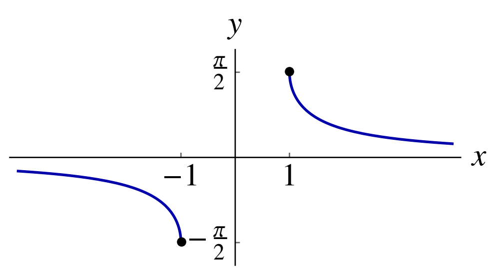
Properties:

(a) The domain is the set of all real numbers except the integer multiples of π; that is:
$$\{x : x \neq k\pi, \quad k \in \mathbb{Z}\}.$$
(b) The range is $(-\infty, -1] \cup [1, \infty)$.
(c) It is an odd function.
(d) It is periodic with fundamental period 2π.
(e) It is differentiable and $\frac{d}{dx} \csc x = -\cot x \csc x$.

Graph: The cosecant function is one-to-one on the interval $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.



Inverse: The inverse of $f(x) = \csc(x)$ on the interval $[-\pi/2, 0) \cup (0\pi/2]$ is $f^{-1}(x) = \text{arccsc}\,(x)$.



(a) The domain of arccsc is $(-\infty, -1] \cup [1, \infty)$ and the range is $[-\pi/2, 0) \cup (0\pi/2]$.

(b) $\text{arccsc}\,(\csc(x)) = x$ for $x \in [-\pi/2, 0) \cup (0, \pi/2]$

(c) $\csc(\text{arccsc}\,(x)) = x$ for $x \in (-\infty, -1] \cup [1, \infty)$

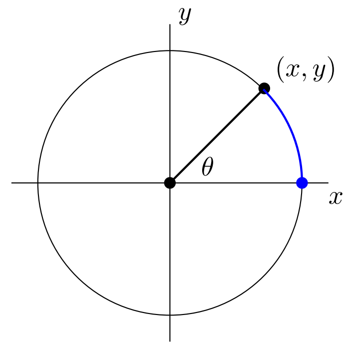
(d) $\frac{d}{dx}\text{arccsc}\,(x) = -\frac{1}{x\sqrt{x^2-1}}$

The cotangent function

Definition: We define the cotangent function by

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}.$$

In terms of the unit circle, form the angle with vertex at the origin, initial side on the positive x-axis, and having angle measure $|\theta|$, rotating the terminal side counterclockwise if $\theta > 0$, and clockwise if $\theta < 0$. Let (x, y) be the coordinates of the point where the terminal side intersects the unit circle $x^2 + y^2 = 1$. Then $\cot(\theta) = \frac{x}{y}$.

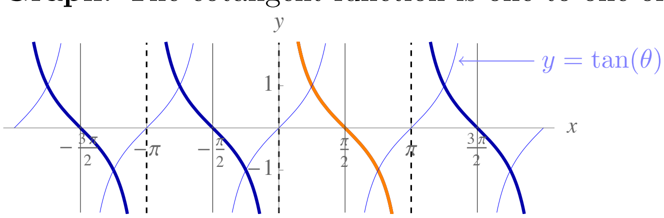


$\cot(\theta) = \frac{x}{y}$

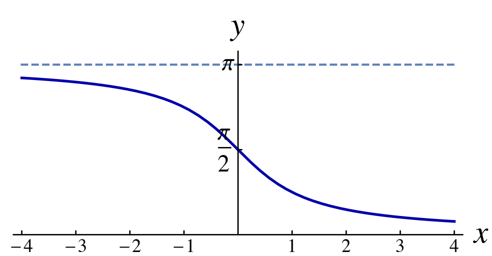
Properties:

(a) The domain is the set of all real numbers except the integer multiples of π; that is:
$$\{x : x \neq k\pi, \quad k \in \mathbb{Z}\}.$$
(b) The range is $(-\infty, \infty)$.
(c) It is an odd function.
(d) It is periodic with fundamental period π.
(e) It is differentiable and $\frac{d}{dx}\cot(x) = -[\csc(x)]^2$.

Graph: The cotangent function is one-to-one on the interval $(0, \pi)$.



Inverse: The inverse of $f(x) = \cot(x)$ on the interval $(0, \pi)$ is $f^{-1}(x) = \operatorname{arccot}(x)$.



(a) The domain of arccot is $(-\infty, \infty)$ and the range is $(0, \pi)$.

(b) $\operatorname{arccot}(\cot(x)) = x$ for all $x \in (0, \pi)$

(c) $\cot(\operatorname{arccot}(x)) = x$ for all $x \in (-\infty, \infty)$

(d) $\frac{d}{dx}\operatorname{arccot}(x) = -\frac{1}{1+x^2}$

Part III: Trigonometric identities

A number of identities are listed below. Many of these identities can be derived from the definitions or from other basic identities. Those that should be memorized will appear in bold face type. The student should know most of the others, or know how to derive them. The student would do well to practice deriving these identities using the hints given in parentheses. With practice, these derivations can be done very quickly, and even mentally.

Let α and β be real numbers. Then:

1. $\boldsymbol{[\sin(\alpha)]^2 + [\cos(\alpha)]^2 = 1}$ (From definitions.*)
2. $[\tan(\alpha)]^2 + 1 = [\sec(\alpha)]^2$ (Obtained from 1.)
3. $1 + [\cot(\alpha)]^2 = [\csc(\alpha)]^2$ (Obtained from 1.)
4. $\sin(-\alpha) = -\sin(\alpha)$ (From definitions.)
5. $\cos(-\alpha) = \cos(\alpha)$ (From definitions.)
6. $\cos(\alpha) = \sin\left(\alpha + \frac{\pi}{2}\right)$ (From graphs.)
7. $\cos(\alpha) = -\sin\left(\alpha - \frac{\pi}{2}\right)$ (From graphs.)
8. $\boldsymbol{\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)}$
9. $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$ (Obtained from 8.)
10. $\boldsymbol{\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)}$
11. $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ (Obtained from 10.)
12. $\tan(\alpha + \beta) = \dfrac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$ (From 8 and 10.)
13. $\tan(\alpha - \beta) = \dfrac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$ (Obtained from 12.)
14. $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$ (Obtained from 8.)
15. $\cos(2\alpha) = [\cos(\alpha)]^2 - [\sin(\alpha)]^2$ (Obtained from 10.)
16. $\cos(2\alpha) = 1 - 2[\sin(\alpha)]^2$ (From 1 and 15.)
17. $\cos(2\alpha) = 2[\cos(\alpha)]^2 - 1$ (From 1 and 15.)

*Also follows from 11 by taking $\alpha = \beta$.

18. $\tan(2\alpha) = \dfrac{2\tan(\alpha)}{1 - [\tan(\alpha)]^2}$ (Obtained from 12.)

19. $\sin(\alpha)\cos(\beta) = \frac{1}{2}\big(\sin(\alpha+\beta) + \sin(\alpha-\beta)\big)$ (From 8 and 9.)

20. $\sin(\alpha)\sin(\beta) = \frac{1}{2}\big(\cos(\alpha-\beta) - \cos(\alpha+\beta)\big)$ (From 10 and 11.)

21. $\cos(\alpha)\cos(\beta) = \frac{1}{2}\big(\cos(\alpha+\beta) + \cos(\alpha-\beta)\big)$ (From 10 and 11.)

22. $\sin(\alpha) + \sin(\beta) = 2\ \sin\frac{\alpha+\beta}{2}\ \cos\frac{\alpha-\beta}{2}$ (From 8 and 9.)

23. $\cos(\alpha) + \cos(\beta) = 2\ \cos\frac{\alpha+\beta}{2}\ \cos\frac{\alpha-\beta}{2}$ (From 10 and 11.)

24. $\sin(\frac{\alpha}{2}) = \pm\sqrt{\dfrac{1-\cos(\alpha)}{2}}$ (Obtained from 16.)

25. $\cos(\frac{\alpha}{2}) = \pm\sqrt{\dfrac{1+\cos(\alpha)}{2}}$ (Obtained from 17.)

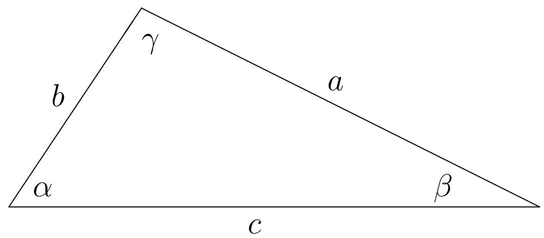
26. $\tan(\frac{\alpha}{2}) = \pm\sqrt{\dfrac{1-\cos(\alpha)}{1+\cos(\alpha)}}$ (From 24 and 25.)

27. $[\sin(\alpha)]^2 = \dfrac{1-\cos(2\alpha)}{2}$ (Obtained from 24.)

28. $[\cos(\alpha)]^2 = \dfrac{1+\cos(2\alpha)}{2}$ (Obtained from 25.)

29. $[\tan(\alpha)]^2 = \dfrac{1-\cos(2\alpha)}{1+\cos(2\alpha)}$ (From 27 and 28.)

Below is drawn an arbitrary triangle. The angles have measure α, β, and γ, and the opposing sides are a, b, and c, respectively.



The Law of Sines:

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

The Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$$

Heron's formula: The area of the triangle is give by the formula

$$\sqrt{p(p-a)(p-b)(p-c)},$$

where $p = \frac{1}{2}(a+b+c)$.