7.4 #6 (from textbook)

$$\vec{x}^{(1)} = \begin{pmatrix} t \\ 1 \end{pmatrix} \qquad \vec{x}^{(2)} = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$$

(a) $W[\vec{x}^{(1)}, \vec{x}^{(2)}] = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = \boxed{t^2}$

(b) The vectors are linearly independent in any interval for which $W[\vec{x}^{(1)}, \vec{x}^{(2)}] = t^2 \neq 0$; that is, for which $t \neq 0$:

$$\boxed{(-\infty, 0) \text{ and } (0, \infty)}$$

(c) Since $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are linearly dependent at $t=0$, we can conclude that they do not form a fund. set of solutions on any interval containing $t=0$. This means that **at least one of the coefficients is not continuous at $t=0$**

Note: Only the boxed text is necessary for the answer.

(d) We assume $P\vec{x} = \vec{x}'$ for $\vec{x} = \vec{x}^{(1)}$ and $\vec{x} = \vec{x}^{(2)}$, so:

$$P\begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad P\begin{pmatrix} t^2 \\ 2t \end{pmatrix} = \begin{pmatrix} 2t \\ 2 \end{pmatrix}$$

$$\Rightarrow \quad P\underbrace{\begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 1 & 2t \\ 0 & 2 \end{pmatrix}}_{X'}$$

Here, X and X' are matrices

Note: $\det X = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = W[\vec{x}^{(1)}, \vec{x}^{(2)}] = t^2$

We are assuming that $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ form a fund. set of solutions, so we are on an interval not containing $t=0$. This means

$$\det X \neq 0 \Rightarrow X^{-1} \text{ exists.}$$

$$X^{-1} = \frac{1}{\det X}\begin{pmatrix} 2t & -t^2 \\ -1 & t \end{pmatrix} = \begin{pmatrix} 2/t & -1 \\ -1/t^2 & 1/t \end{pmatrix}$$

$$PX = X' \Rightarrow P = X'X^{-1} = \begin{pmatrix} 1 & 2t \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2/t & -1 \\ -\frac{1}{t^2} & 1/t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{2}{t^2} & \frac{2}{t} \end{pmatrix}$$

So the system is:

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{2}{t^2}x_1 + \frac{2}{t}x_2 \end{aligned}}$$

Note that two coefficients are not continuous at $t=0$.